Methodologies of predictions and their quality

A. L. Semenov, An. A. Muchnik

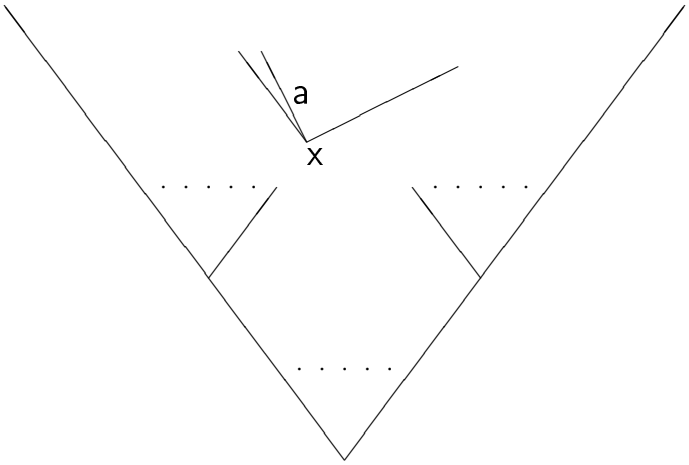
Council of Cybernetics at Computing Center
of Russian Academy of Sciences

The world we investigate $=$
$=$ Tree of states with a measure μ on it.

We are given $\mu_1, \ldots, \mu_k$ — measures on the tree (hypotheses).

We presuppose that one of μ_i corresponds to the real world (but we do not know which μ_i).

Our goal is to produce a hypothesis that should be as close to the real μ_i as possible.
How can we measure the closeness?



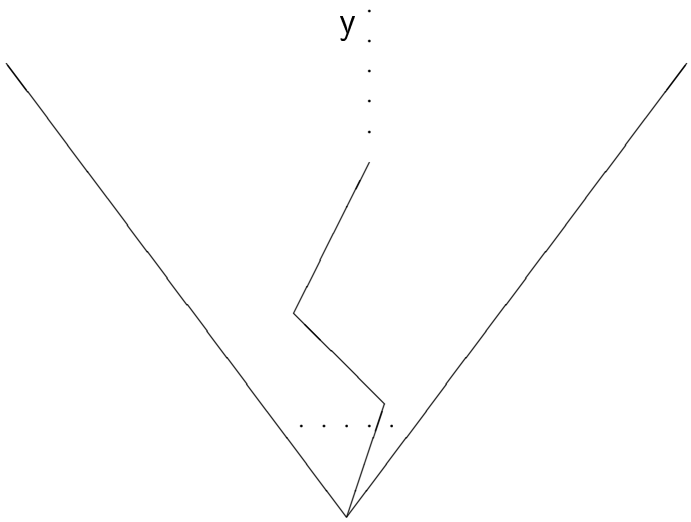
Let tree of states be A^*.

Every hypothesis μ for any node x sets a probability distribution on A:

$$p(a) = \mu(xa|x) = \mu(xa)/\mu(x).$$

Our characteristic of closeness between two such probability distributions is Hellinger distance:

$$\rho(p,q) \;=\; \sum\nolimits_a (\sqrt{p(a)} - \sqrt{q(a)})^2 \;= \\ = 2 - 2\sum\nolimits_a \sqrt{p(a)q(a)}.$$



Characteristic of closeness between hypothesis ν and μ along the path y in the tree is

$$r(\mu, \nu, y) = \sum\nolimits_k \rho(\, \mu(\cdot|y_{1:k}),\ \nu(\cdot|y_{1:k})).$$

Global characteristic of closeness between hypothesis ν and hypothesis μ is

$$R_\alpha(\mu, \nu) = \ \mathbf{E}[\exp(\alpha r(\mu, \nu, y))].$$

$\mathbf{E}$ — probabilistic average by measure corresponding to the real world.
α — an auxiliary parameter.

For a given k the quality of methodology producing ν for $\mu_1, \ldots, \mu_k$ is measured by a positive real number

$$\sup\{\alpha\colon \forall \mu_1, \ldots, \mu_k \; \forall i \; (\mu_i \text{ is real} \to R_\alpha(\mu_i, \nu) < \infty)\}.$$

If all μ_i and ν are computable, then previous expression is equal to:

$$\sup\{\alpha\colon \forall \mu_1, \ldots, \mu_k \; \forall i$$
$$(\mu_i \text{ is real} \to \forall y \; r(\mu_i, \nu, y) < \frac{\ln 2}{\alpha} d_{\mu_i}(y) + O(1))\},$$

where $d_\mu(y)$ — randomness deficiency of y relative to μ.

Methodology of Ray Solomonoff

$$\nu = \frac{1}{k} \sum_{i=1}^{k} \mu_i, \quad \max \alpha = \frac{1}{2} \;\; (\forall k \geq 2).$$

Best-buy methodology

$\nu(xa|x) = \mu_i(xa|x)$, where $\mu_i(x) = \max_j \mu_j(x)$,

$$\sup \alpha \leq \frac{1}{2} \;\; (\forall k \geq 2).$$

Optimal methodology for $k = 2$

$$\sup \alpha = 1.$$

Theorem.

1. Let $k \geq 2$. For any methodology $\sup \alpha \leq 1$.

2. Let $k = 2$. Given natural number n we can effectively construct a methodology such that $\sup \alpha \geq 1 - \frac{1}{n}$.

Problem: does there exist one methodology such that $\max \alpha = 1$?