Constructive series, descriptive complexity, simple and hypersimple sets

Alexei Semenov Andrej Muchnik

Institute of New Technologies
Moscow

$$\log^{[1]} x = \log x, \quad \log^{[i+1]} x = \log(\log^{[i]} x)$$

$$a_m^0 = \frac{1}{m^2}, \quad a_m^1 = \frac{1}{m(\log m)^2}, \quad a_m^2 = \frac{1}{m \log m (\log^{[2]} m)^2}, \quad \dots$$

$$b_m^0 = \frac{1}{m}, \quad b_m^1 = \frac{1}{m \log m}, \quad b_m^2 = \frac{1}{m \log m \log^{[2]} m}, \quad \dots$$

$$\infty > \sum a_m^0, \quad \sum a_m^1, \quad \sum a_m^2, \quad \dots \ ? \ \dots \quad \sum b_m^2, \quad \sum b_m^1, \quad \sum b_m^0 = \infty$$

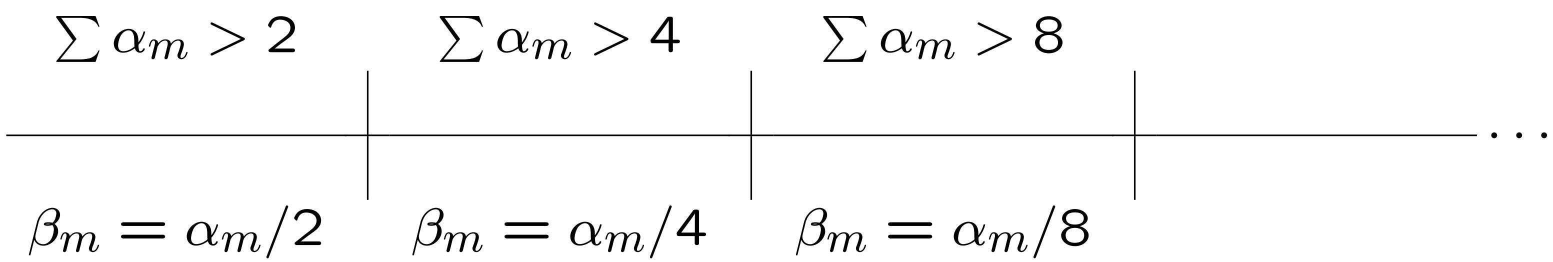
$$a_m^n = \frac{1}{m \log^{[1]} m \ldots \log^{[n-1]} (\log^{[n]} m)^2}\,, \; b_m^n = \frac{1}{m \log^{[1]} m \ldots \log^{[n-1]} \log^{[n]} m}\,,$$

$$b_m^n / a_m^n = \log^{[n]} m$$

Computable positive series

(w.l.o.g., members of series are of the form 2^{-n})

$$\sum \alpha_m = \infty \quad \Rightarrow \quad \exists \beta_m \ : \ \frac{\beta_m}{\alpha_m} \to 0 \ \& \ \sum \beta_m = \infty$$



Computable positive series

$$\sum \alpha_m < \infty \text{ effectively} \quad \Rightarrow \quad \exists \beta_m : \frac{\beta_m}{\alpha_m} \to \infty \ \& \ \sum \beta_m < \infty \text{ effectively}$$

$\sum^\infty \alpha_m < 1/4$	$\sum^\infty \alpha_m < 1/16$	$\sum^\infty \alpha_m < 1/64$	$\ldots$
$\beta_m = \alpha_m \cdot 2$	$\beta_m = \alpha_m \cdot 4$	$\beta_m = \alpha_m \cdot 8$	

Computable positive series

Theorem 1.

$$\exists \alpha_m \ \sum \alpha_m < \infty \quad \& \quad \left(\forall \beta_m \ \ \frac{\beta_m}{\alpha_m} \to \infty \ \Rightarrow \ \sum \beta_m = \infty \right)$$

$$\forall \alpha_m \ \sum \alpha_m < \infty \quad \Rightarrow \quad \left(\exists \beta_m \ \ \sum \beta_m < \infty \ \& \ \ \text{sup} \frac{\beta_m}{\alpha_m} = \infty \right)$$

Series computably approximated from below

(enumerable series)

Definition

$\nu(x)$ is *enumerable series* iff $\exists$ computable function $\nu(x,t) \geqslant 0$

$$\forall x \;\; \nu(x) = \lim_{t \to \infty} \nu(x,t)$$
$$\forall x, t \;\; \nu(x, t+1) \geqslant \nu(x,t)$$
$$\forall t \;\; \{x \mid \nu(x,t) \neq 0\} \text{ is finite}$$

Theorem 2 (Levin).

$$\exists \mu(x) \;:\; \sum \mu(x) \leqslant 1 \;\;\&\;\; \forall \nu(x) \Big(\sum \nu(x) \leqslant 1 \;\Rightarrow\; \exists C \, \forall x \; \mu(x) \geqslant \nu(x) \cdot 2^{-C} \Big)$$

This μ is <u>unique</u> up to a multiplicative constant

Series computably approximated from below

Theorem 2 (Levin).

$$\exists\mu(x)\ :\ \sum\mu(x)\leqslant 1\ \&\ \forall\nu(x)\Big(\sum\nu(x)\leqslant 1\ \Rightarrow\ \exists C\,\forall x\ \mu(x)\geqslant\nu(x)\cdot 2^{-C}\Big)$$

Fact I.
All computable enumerations of enumerable series are m-reducible to some universal computable enumeration $n\mapsto\nu_n$.

Fact II.
Each enumerable series ν can be effectively transformed into an enumerable series ν' such that

$\sum\nu'(x)\leqslant 1$,

$\sum\nu(x)\leqslant 1\ \Rightarrow\ \forall x\,\nu'(x)=\nu(x)$.

$$\mu(x)\rightleftharpoons\sum_n 2^{-n}\nu'_n(x)$$

Bounds for μ

Theorem 3. (cf. Marandjian)
For any partial computable function γ

$$\exists C \qquad \forall x \in \mathrm{Dom}(\gamma)\ \ \mu(x) \leqslant \gamma(x) \quad \Rightarrow \quad \forall x \in \mathrm{Dom}(\gamma)\ \ \gamma(x) \geqslant 2^{-C}.$$

Theorem 4.
There exists a computable series α such that
$\forall x \quad \alpha(x) \leqslant \mu(x)$,
$\exists^{\infty} x\ \alpha(x) = \mu(x)$.

Obviously, Theorem 4 implies Theorem 1.

Proof of Theorem 4

1					
2					
$\vdots$					
i	$\beta(x_i^1) = 2^{-i}$				
$\vdots$					

Proof of Theorem 4

1					
2					
$\vdots$					
i	$\beta(x_i^1) = 2^{-i}$ $\mu(x_i^1) > 2^{-i}$	$\beta(x_i^2) = 2^{-i}$			
$\vdots$					

Proof of Theorem 4

1					
2					
$\vdots$					
i	$\beta(x_i^1) = 2^{-i}$ $\mu(x_i^1) > 2^{-i}$	$\beta(x_i^2) = 2^{-i}$ $\mu(x_i^2) > 2^{-i}$	$\ldots$	$\beta(x_i^j) = 2^{-i}$	
$\vdots$					

Proof of Theorem 4

$\{x : \mu(x) \leqslant \beta(x)\}$ is infinite

β is computable

$\sum \beta(x) \leqslant \sum_x \mu(x) + \sum_i 2^{-i} \leqslant 1 + 1$

$\exists c \forall x \;\; \mu(x) > \beta(x) 2^{-c}$

$\{x : \mu(x) \leqslant \beta(x) 2^{-c}\}$ is empty

$\exists d \in [0, c]$

$$\{x : \mu(x) \leqslant \beta(x) 2^{-d}\} \text{ is infinite}$$
$$\{x : \mu(x) \leqslant \beta(x) 2^{-(d+1)}\} \text{ is finite}$$

Proof of Theorem 4

$$\exists^\infty x \;\; \mu(x) = \beta(x)2^{-d}$$

$$\forall^\infty x \;\; \mu(x) \geqslant \beta(x)2^{-d}$$

$$\alpha(x) = \min\{\mu(x), \beta(x)2^{-d}\}$$

Bounds for μ

Theorem 5.
For any computable series $\alpha(x)$

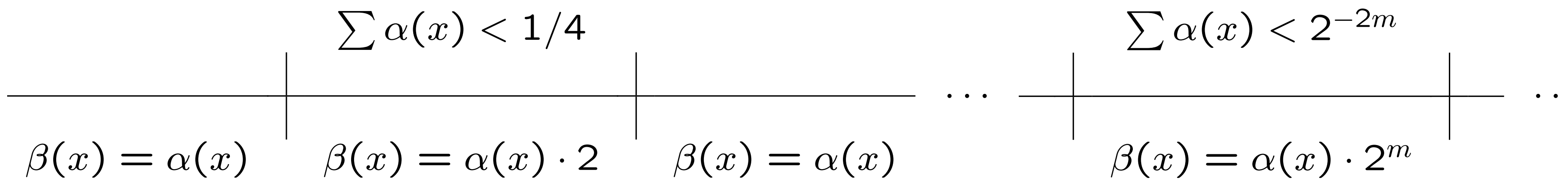
$$\forall x\ \alpha(x) \leqslant \mu(x), \quad \exists^{\infty} x\ \alpha(x) = \mu(x)$$

$$\Downarrow$$

the set $S = \{x : \alpha(x) < \mu(x)\}$ is hypersimple

(An enumerable set S with infinite complement is *hypersimple* if for any computable sequence $0 < j_1 < j_2 < \ldots$ there exists k such that $[j_k, j_{k+1}) \subset S$.)

Proof of Theorem 5



β is computable

$\sum \beta(x) \leqslant \sum_x \alpha(x) + \sum_m 2^{-2m} \cdot 2^m \leqslant \sum_x \mu(x) + \sum_m 2^{-m} \leqslant 1 + 1$

$\exists c\, \forall x \;\; \mu(x) \geqslant \beta(x) 2^{-c}$

The whole segment corresponding to $m = c$ is included in S.

Descriptive complexity

Plain entropy (Kolmogorov, 1965) KS is the minimal length of a code word with respect to an optimal coding.

A coding is called *prefix* if no code word is a beginning of any other code word. (Useful for dividing a message flow.)

Prefix entropy (Levin, 1970) KP is the minimal length of a code word with respect to an optimal prefix coding.

Plain entropy

Theorem.
There exists a computable function f such that
$\forall x \quad f(x) \geqslant KS(x)$,
$\exists^{\infty} x\ f(x) = KS(x)$.

$(f(x) = \log_2(x) + O(1))$

Theorem 6.
For any computable function f

$$\forall x\ f(x) \geqslant KS(x), \quad \exists^{\infty} x\ f(x) = KS(x)$$

$$\Downarrow$$

the set $S = \{x : f(x) > KS(x)\}$ is simple

Prefix entropy

Theorem (Levin).
$KP(x) = -\log_2 \mu(x) + O(1)$

Theorem 7 (Solovay).
There exists a computable function f such that
$\forall x \quad f(x) \geqslant KP(x)$,
$\exists^\infty x\ f(x) = KP(x)$.

Theorem 8 (Solovay).
For any computable function f

$$\forall x\ f(x) \geqslant KP(x), \quad \exists^\infty x\ f(x) = KP(x)$$

$$\Downarrow$$

the set $S = \{x : f(x) > KP(x)\}$ is hypersimple