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# Weakly symmetric spaces

Weakly symmetric spaces (w.s.s.) were introduced by A.Selberg in connection with his celebrated trace formula. In his definition, w.s.s. are Riemannian homogeneous manifolds with an additional isometry of a special kind generalizing symmetry in the classical case. From the properties of this isometry it follows that the manifold is commutative, i.e., its algebra of invariant differential operators is commutative. In a joint paper with the speaker, E.B.Vinberg carried over the definition of w.s.s. to the theory of algebraic transformation groups and proved that algebraic w.s.s. of reductive groups are exactly affine homogeneous spherical varieties. As a consequence, it became clear that, again for reductive groups, the initial definition of w.s.s. is equivalent to the commutativity of the Riemannian homogeneous manifold in question.

More precisely, let $G$ be a connected complex reductive algebraic group, $H \subset G$ a closed reductive subgroup, $X = G/H$ the corresponding homogeneous space, and $\Delta \subset G \times G$ the diagonal subgroup acting on $X \times X$. Then $X$ is called an algebraic w.s.s. if $X$ is acted on by an extension $\hat{G} = \langle G, s \rangle$ of index 2 of $G$ in such a way that the mapping $(x, y) \to (sy, sx)$ induces the identity on the categorical quotient $(X \times X)//\Delta$. Let $\sigma$ be the automorphism of $G$ defined by $\sigma(g) = sgs^{-1}$. One can arrange that $\sigma(H) = H$. Then $\sigma(g) \in Hg^{-1}H$ for $g \in G$ generic. Following the pattern of classical harmonic analysis, one concludes that an algebraic w.s.s. is spherical. The converse is also true, but the proof is rather involved, using the classification of spherical homogeneous spaces. In the talk, we will review the proof and give some new applications to the multiplicity free actions in complex analytic context.

Let $D$ be a connected complex manifold, $\mathcal{O}(D)$ the space of holomorphic functions on $D$ and $K$ a maximal compact subgroup of $G$ acting by holomorphic transforms on $D$. If there exists an antiholomorphic involution $\mu : D \to D$ such that

$$\mu(p) \in Kp \quad \text{for all } p \in D \tag{1}$$

then, by a result of J.Faraut and E.G.F.Thomas, $\mathcal{O}(D)$ is a multiplicity free $K$-module. In some special cases, this sufficient condition turns out to be necessary. One of these situations is considered below. Fix a Weyl involution $\theta$ of $K$ and assume that $\mu : D \to D$ is a $\theta$-equivariant antiholomorphic involution, i.e.,

$$\mu(kp) = \theta(k)\mu(p) \quad \text{for all } p \in D. \tag{2}$$

For example, such a $\mu$ always exists if $D$ is a complex vector space with a linear action of $K$. The following results are obtained jointly with A.Püttmann.

**Theorem 1.** Suppose $D$ is holomorphically separable and there is an antiholomorphic involution $\mu : D \to D$ satisfying (2). Let $D' \subset D$ be a $K$-invariant domain. Then the following assertions are equivalent:
(i) $\mathcal{O}(D')$ is multiplicity free;
(ii) $\mathcal{O}(D)$ is multiplicity free;
(iii) $\mu$ has property (1);
(iv) there exists an antiholomorphic involution $\mu' : D' \to D'$ for which (1) holds with $D, \mu$ replaced by $D', \mu'$.

**Theorem 2.** Let $D$ be a $K$-invariant domain in an affine homogeneous space $X = G/H$. Then $\mathcal{O}(D)$ is a multiplicity free $K$-module if and only if there exists an antiholomorphic involution $\mu : D \to D$ satisfying (1). In fact, if $\mathcal{O}(D)$ is multiplicity free then $X$ is spherical and $\mu$ can be chosen so that both (1) and (2) are fulfilled.