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Reynolds Operator

Let k be a commutative ring with unit. A k-module E can be considered as a functor of k-modules over the category of commutative k-algebras, which we will denote by $\mathbf{E}$, by defining $\mathbf{E}(B) := E \otimes_k B$. If F and G are functors of k-modules, we will denote by $\mathbf{Hom}_k(F, H)$ the functor of k-modules

$$\mathbf{Hom}_k(F, H)(B) := \mathrm{Hom}_B(F_{|B}, H_{|B})$$

where $F_{|B}$ is the functor F restricted to the category of commutative B-algebras. The functor $F^* := \mathbf{Hom}_k(F, \mathbf{k})$ is said to be a dual functor. For example, $\mathbf{E}$, $\mathbf{E}^*$ and $\mathbf{Hom}_k(\mathbf{E}, \mathbf{E}')$ are dual functors (see [A, 1.10]).

An affine k-monoid $G = \mathrm{Spec}\, A$ can be considered as a functor of monoids over the category of commutative k-algebras: $G^{\cdot}(B) := \mathrm{Hom}_{k-sch}(\mathrm{Spec}\, B, G)$. A functor of G-modules (respectively of $\mathbf{A}^*$-modules) is a functor of k-modules endowed with a linear action of $G^{\cdot}$ (respectively of $\mathbf{A}^*$). In this paper, we prove the following theorem.

Theorem 1. *The category of dual functors of G-modules is equivalent to the category of dual functors of $\mathbf{A}^*$-modules.*

We prove that an affine k-group $G = \mathrm{Spec}\, A$ is semisimple if and only if $\mathbf{A}^* = \mathbf{k} \times \mathbf{B}^*$ as functors of k-algebras (and the first projection $A^* \to k$ is an element of A). If G is semisimple there exists an isomorphism $A^* = k \times B^*$ such that the first projection $A^* \to k$ is the unit of A. The linear form $w_G := (1, 0) \in k \times B^* = A^*$, which will be referred to as the *invariant integral* of G. We prove the following theorem.

Theorem 2. *Let $G = \mathrm{Spec}\, A$ be a semisimple k-group and let $w_G \in A^*$ be the invariant integral of G. Let F be a dual functor of G-modules. It holds that:*

1. $F^G = w_G \cdot F$.
2. *F splits uniquely as a direct sum of F^G and another subfunctor of G-modules, explicitly*

$$F = w_G \cdot F \oplus (1 - w_G) \cdot F.$$

We call the projection $F \to F^G = w_G \cdot F$ the Reynolds operator. The previous theorem still holds for any separated functor of $\mathbf{A}^*$-modules. More generally, for every functor of G-modules H, we prove that there exists the maximal separated G-invariant quotient of H and that the dual of this quotient is H^{*G}. Moreover, when H is a dual functor, the quotient morphism is the Reynolds operator.

Let R be a G-algebra and let E and V be two RG-modules. In [C] a Reynolds operator is defined on $\mathrm{Hom}_R(E, V)$, generalizing some results of Magid (see [M]). This last result is a particular case of the previous theorem.

In the final example we prove the main results of [F] about generalized Ω-process.

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