M. Brion
Institut Fourier
Grenoble, France
Michel.Brion@ujf-grenoble.fr

Anti-affine algebraic groups

In this talk, we introduce and study the class of groups of the title. We say that a group scheme G of finite type over a field k is anti-affine, if $\mathcal{O}(G) = k$. Then G is known to be smooth, connected and commutative. Examples include, of course, all abelian varieties, but also their universal vector extensions (in characteristic zero only) and certain semi-abelian varieties.

The class of anti-affine groups and the class of affine (or, equivalently, linear) group schemes play complementary roles in the structure of group schemes over fields. Indeed, any connected group scheme G, of finite type over k, has a largest anti-affine subgroup scheme G_{ant}. This subgroup is contained in the centre of G, and the quotient G/G_{ant} is the affinization of G, i.e., the universal affine quotient group scheme. Also, G has a medest normal connected affine subgroup scheme G_{aff} such that G/G_{aff} is an abelian variety. This yields the *Rosenlicht decomposition*: $G = G_{\mathrm{aff}}\, G_{\mathrm{ant}}$ and $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ contains $(G_{\mathrm{ant}})_{\mathrm{aff}}$; moreover, the quotient group scheme $(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})/(G_{\mathrm{ant}})_{\mathrm{aff}}$ is finite.

Affine group schemes have been extensively investigated, but little seems to be known about their anti-affine counterparts; they only appear implicitly in work of Rosenlicht and Serre. Here we present some fundamental properties of anti-affine groups, which reduce their structure to that of abelian varieties.

Our main result classifies anti-affine algebraic groups G over an arbitrary field. In positive characteristics, G is a semi-abelian variety, parametrized by a pair (A, Λ) where A is an abelian variety and Λ is a sublattice of the group of geometric points of A, stable unde the absolute Galois group. The classification is a bit more complicated in characteristic zero: the parameters are then triples (A, Λ, V) where A and Λ are as above, and V is a subspace of the Lie algebra of A. In both cases, A is the dual of the abelian variety G/G_{aff}.

As a consequence, every anti-affine group over a finite field is an abelian variety. Combined with the Rosenlicht decomposition, it follows that any connected group scheme G over a finite field has a decomposition $G = G_{\mathrm{aff}}G_{\mathrm{ab}}$, where G_{ab} is the largest abelian subvariety of G; moreover, $G_{\mathrm{aff}} \cap G_{\mathrm{ab}}$ is finite. For algebraic groups, this result is due to Arima.

Our classification also implies a structure result for connected algebraic groups G over any perfect field k of positive characteristic, namely, the decomposition $G = G_{\mathrm{uni}}\, S$ where $G_{\mathrm{uni}} \subset G_{\mathrm{aff}}$ denotes the medest normal connected subgroup such that $G_{\mathrm{aff}}/G_{\mathrm{uni}}$ is a torus, and $S \subset G$ is a semi-abelian subvariety; moreover, $G_{\mathrm{uni}} \cap S$ is finite. If k is algebraically closed, then the group G_{uni} is generated by all connected unipotent subgroups of G.

Another application concerns Hilbert's fourteenth problem in its algebro-geometric formulation: does any quasi-affine variety have a finitely generated coordinate ring? The answer is known to be negative, the first counterexample being due to Rees. Here we obtain many counterexamples, namely, all $\mathbb{G}_m$-torsors associated to ample line bundles over anti-affine, non-complete groups.