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Computing maximal Abelian dimensions in upper-triangular matrix algebras

The topic which is dealt in this paper is the maximal abelian dimension of a given finite-dimensional Lie algebra $\mathfrak{g}$, that is, the maximum among the dimensions of the abelian Lie subalgebras of $\mathfrak{g}$.

More concretely, this maximum is computed for the Lie algebra $\mathfrak{h}_n$, formed by all the $n\times n$ upper-triangular matrices. In this way, every vector in $\mathfrak{h}_n$ can be expressed as follows:

$$h_n(x_{r,s}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{nn} \end{pmatrix}$$

Starting from this expression for the vectors in $\mathfrak{h}_n$, the following basis can be obtained: $\mathcal{B} = \{X_{i,j} = h_n(x_{r,s}) \mid 1 \leq i \leq j \leq n\}$, where:

$$x_{r,s} = \begin{cases} 1, & \text{if } (r,s) = (i,j), \\ 0, & \text{if } (r,s) \neq (i,j). \end{cases}$$

Therefore, the dimension of this algebra is:

$$\dim(\mathfrak{h}_n) = d_{\mathfrak{h}_n} = \frac{n(n+1)}{2},$$

and the nonzero brackets with respect to the basis $\mathcal{B}$ are:

$[X_{i,j}, X_{j,k}] = X_{i,k}, \quad \forall i = 1 \ldots n-2,\, \forall j = i+1 \ldots n-1,\, \forall k = j+1 \ldots n.$
$[X_{i,i}, X_{i,j}] = X_{i,j}, \quad \forall j > i.$
$[X_{k,i}, X_{i,i}] = X_{k,i}, \quad \forall k < i.$

Apart from other reasons related to Physics, our interest for studying the Lie algebras $\mathfrak{h}_n$ lies in the fact which every finite-dimensional solvable Lie algebra is isomorphic to a Lie subalgebra in some Lie algebra $\mathfrak{h}_n$ [1, Proposition 3.7.3]. Therefore, the computation of the maximal abelian dimension for $\mathfrak{h}_n$ can be considered a first step to study the maximal abelian dimension of any given finite-dimensional solvable Lie algebra.

The present paper continues the authors' previous paper [2] in which some properties of the maximal abelian dimension were studied for the algebra $\mathfrak{h}_n$ and a value for its maximal abelian dimension was conjectured:

Conjecture. Fixed and given $n \in \mathbb{N} \setminus \{1\}$, the maximal abelian dimension of $\mathfrak{h}_n$ is:

$$\mathcal{M}(\mathfrak{h}_n) = \begin{cases} n, & \text{if } n < 4, \\ k^2 + 1, & \text{if } n = 2k,\ n \geq 4, \\ k^2 + k + 1, & \text{if } n = 2k+1,\ n \geq 4. \end{cases}$$

This conjecture was achieved starting from an algorithmic procedure to compute abelian subalgebras in the Lie algebra $\mathfrak{h}_n$. In fact, this conjecture was already proved for the particular cases $n = 2$ and $n = 3$ in [2].

In this paper, we show a proof for the previously commented conjecture for all $n \in \mathbb{N} \setminus \{1\}$. To do it, two lemmas will be proved and applied in order to prove the veracity of the conjecture. To get the proof, the vectors in a given basis of $\mathfrak{h}_n$ have to be distinguished between *main vectors* and *non-main* ones for a given basis of the subalgebra. Such a distinction is based on writing each vector in the basis of the subalgebra as a linear combination of the basis of $\mathfrak{h}_n$; then these coefficients are written as the rows in a matrix and the vectors corresponding to the pivot positions of its echalon form are the main vectors.

References

[1] V.S. Varadarajan, Lie Groups, Lie Algebras and Their Representations, Springer, New York, 1984.

[2] M. Ceballos, J. Núñez, A.F. Tenorio, Obstructions to represent abelian Lie subalgebras in the Lie algebra of upper-triangular matrices. *First French-Spanish Congress of Mathematics*, Saragossa (Spain), July 2007.