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Commutative cocycles and algebras with antisymmetric identities

Algebras with one of the following identities are considered:
$[[t_1,t_2],t_3]+[[t_2,t_3],t_1]+[[t_3,t_1],t_2]=0$ (Lie-Admissible),
$[t_1,t_2]t_3+[t_2,t_3]t_1+[t_3,t_1]t_2=0$ (0-Lie-Admissible, shortly 0-Alia),
$\{[t_1,t_2],t_3\}+\{[t_2,t_3],t_1\}+\{[t_3,t_1],t_2\}=0$ (1-Lie-admissible, shortly 1-Alia), where $[t_1,t_2]=t_1t_2-t_2t_1$ and $\{t_1,t_2\}=t_1t_2+t_2t_1$. For an algebra $A=(A,\circ)$ with multiplication $\circ$ denote by $A^{(q)}$ an algebra with vector space A and multiplication $a\circ_q b=a\circ b+q\,b\circ a$.

Theorem 1. *Any algebra with a skew-symmetric identity of degree* 3 *is (anti)-isomorphic to one of the following algebras:*

- *Lie-admissible algebra*
- 0-*Alia algebra*
- 1-*Alia algebra*
- *algebra of the form $A^{(q)}$ for some left-Alia algebra A and $q\in K$, such that $q^2\neq 0,1$.*

Any right (left) Alia algebra is anti-isomorphic to its opposite algebra, left (right) Alia Algebra.

For anti-commutative algebra $(A,\circ)$ call a bilinear map $\psi : A\times A\to A$ *commutative cocycle*, if

$$\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b)=0,$$
$$\psi(a,b)=\psi(b,a),$$

for any $a,b,c\in A$. Algebra with identities

$$[a,b]\circ c+[b,c]\circ a+[c,a]\circ b=0$$

$$a\circ[b,c]+b\circ[c,a]+c\circ[a,b]=0$$

is called *two-sided Alia.*

Theorem 2. *For any anti-commutative algebra $(A,\circ)$ with commutative cocycle ψ an algebra $(A,\circ_\psi)$, where $a\circ_\psi b=a\circ b+\psi(a,b)$, is 1-Alia. Conversely, any 1-Alia algebra is isomorphic to algebra of a form $(A,\circ_\psi)$ for some anti-commutative algebra A and some commutative cocycle ψ. Moreover, if $(A,\circ)$ is Lie algebra with commutative cocycle ψ, then $(A,\circ_\psi)$ is two-sided Alia and, conversely, any two-sided Alia algebra is isomorphic to algebra of a form $(A,\circ_\psi)$ for some Lie algebra A and commutative cocycle ψ.*

Theorem 3. *Let L be classical Lie algebra over a field of characteristic $p\neq 2$. Then it has non-trivial commutative cocycles only in the following cases $L=sl_2$ or $p=3$.*

Standard construction of q-Alia algebras. Let $(U,\cdot)$ be associative commutative algebra with linear maps $f,g:U\to U$. Denote by $\mathcal{A}_q(U,\cdot,f,g)$ an algebra defined on a vector space U by the rule

$$a\circ b=a\cdot f(b)+g(a\cdot b)-q\,f(a)\cdot b.$$

Then $\mathcal{A}_q(U,\cdot,f,g)$ is q-Alia.

Example. $(\mathbf{C}[x],\star)$ under multiplication $a\star b=\partial(a)\partial^2(b)$ is 1-Alia and simple.

Example. $(\mathbf{C}[x],\star)$, where $a\star b=\partial^3(a)b+4\partial^2(a)\partial(b)+5\partial(a)\partial^2(b)+2a\,\partial^3(b)$, is 0-Alia and simple. It is exceptional 0-Alia algebra.

Example. Let $(\lambda_{i,j})$ be symmetric matrix. Then $(\mathbf{C}[x_1,\ldots,x_n],\star)$, where $a\star b=\sum_{\lambda_{i,j}}(\partial_i(a)\partial_j(b)+\partial_i\partial_j(a)b/2)$ is 0-Alia. It is simple iff the matrix $(\lambda_{i,j})$ is non-degenerate.

Example. Let m be positive integer and $A = (\mathbf{C}[x], \star)$ an algebra with multiplication $a \star b = a\partial^m(b) - q\partial^m(a)b + q\,\partial^m(ab)$ Then A is q-Alia and simple.

Let s_k be standard skew-symmetric polynomial,

$$s_k = \sum_{\sigma \in Sym_k} sign\,\sigma\, t_{\sigma(1)} \cdots t_{\sigma(k)}.$$

For a skew-symmetric polynomial f an anti-commutative algebra $(A, \circ)$ is called *f-Lie* if it satisfies the identity $f = 0$. Call it *minimal f-Lie* if $f = 0$ is minimal identity that does not follow from anti-commutativity identity. For example, any Lie algebra is s_4-Lie. There exist interesting examples of simple minimal s_4-Lie algebras.

Theorem 4. *Let U be an associative commutative algebra with derivations D_1, D_2. Then $(U, D_1 \wedge D_2)$ is s_4-Lie. This algebra is Lie if differential system $\{D_1, D_2\}$ is in involution.*

Theorem 5. *Let U be an associative commutative algebra with derivation D. Then $(U, id \wedge D^2)$ is s_4-Lie.*

Example. Algebra with base $\{e_i, i \geq -1\}$ and multiplication

$$e_i \circ e_j = (i - j)(i + j + 3)e_{i+j}$$

is minimal s_4-Lie and simple.

Theorem 6. *Let A be s_d-Lie, where $d = 3$ or $d = 4$. If f is a skew-symmetric polynomial of degree $\geq d$, then A is f-Lie.*