Yu. V. Dzyadyk
International Center for Information Technologies and Systems
Kyiv, Ukraine
iurius@mail.ru

Induced representations on symmetric and spherical spaces

I have 4 aims of this report.

A. To remind the main results of the theory of the induced representations over symmetric spaces – Satake schemes, localization theorem, reduction theorem, local subgroups G_α, $G_{\alpha\beta}$, structure of subgroup M, the rôle of the Iwasawa decomposition, Bruhat decomposition of the symmetric spaces etc.

History: É. Cartan (1929), A. Kirillov (1957, 1960), I. Satake (1960), M. Sugiura (1962), I. Gel'fand – I. Bernšteĭn – S. Gel'fand (1974), Yu. Dzyadyk (1975).

B. To establish some landmarks in the history of spherical spaces: Yu. Dzyadyk (1975), M. Krämer (1978, published in 1979), E. Vinberg and B. Kimel'fel'd (1978), I. Mikityuk (1985), M. Brion (1985) etc.

C. Some addenda to the theory of spherical spaces. Exhaustive classification of spherical spaces. Kac schemes of some spherical spaces.

D. Some results and problems about induced representations over spherical spaces.

Part A. Induced representations over symmetric spaces.

§1. Let us introduce usual notations:

G is a connected, simply-connected compact Lie group,

σ is an involutory automorphism of G,

K is the connected subgroup of points fixed by σ,

G/K is a symmetric space, $o = \{K\} \in G/K$ is a pole of G/K,

$M \subset K$ is the stationary subgroup of a point of general position in a neighborhood of o,

$\mathfrak{g}$, $\mathfrak{k}$, ..., $\mathfrak{m}$ are the Lie algebras corresponding to the groups $G, K, \dots, M$, respectively,

$\mathfrak{h}$ is a σ-invariant Cartan subalgebra in $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra in $\mathfrak{m}$,

$\mathfrak{a} = \mathfrak{h}_- = \{h \in \mathfrak{h} : \sigma(h) = -h\}$, $p = \dim \mathfrak{a} = \operatorname{rank} G/K$, $\mathfrak{h}_+ = \{h \in \mathfrak{h} : \sigma(h) = h\} = \mathfrak{h} \cap \mathfrak{k} = \mathfrak{h} \cap \mathfrak{m}$,

$\mathfrak{g}^\alpha \subset \mathfrak{g}$ is a root subspace, corresponding to root α, $\mathfrak{n} = \mathfrak{g}^+ = \sum \mathfrak{g}^\alpha_{\alpha>0}$, $\mathfrak{m}^+ = \mathfrak{m} \cap \mathfrak{g}^+$,

$U(\mathfrak{g})$, $U(\mathfrak{k})$, $\dots, U(\mathfrak{k}_\alpha)$ are universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{k}$, ..., $\mathfrak{k}_\alpha$, respectively.

§2. Compact cyclicity of the highest weight vectors. *Let λ be any highest weight of the algebra $\mathfrak{g}$, and let v_λ be a highest weight vector of the module V_λ. Then vector v_λ is $\mathfrak{k}$–cyclic in V_λ:*

$$U(\mathfrak{k})v_\lambda = U(\mathfrak{k})U(\mathfrak{a})U(\mathfrak{n})v_\lambda = U(\mathfrak{g})v_\lambda = V_\lambda. \tag{1}$$

§3. Localization. Let α be a root of the algebra $\mathfrak{g}$, and $\sigma(\alpha) \neq \alpha$. We let $\mathfrak{g}_\alpha$ denote the smallest subalgebra in $\mathfrak{g}$ which is invariant under the involution σ and containes the root subspace $\mathfrak{g}^\alpha$. We set $\mathfrak{k}_\alpha = \mathfrak{k} \cap \mathfrak{g}_\alpha$, $\mathfrak{h}_\alpha = \mathfrak{h} \cap \mathfrak{g}_\alpha$.

The symmetric pair $(\mathfrak{g}_\alpha, \mathfrak{k}_\alpha)$ is isomorphic to one of the three pairs $(\mathfrak{su}_2, \mathfrak{so}_2)$, $(\mathfrak{su}_3, \mathfrak{u}_2)$ or $(\mathfrak{so}_4, \mathfrak{so}_3)$.

§4. Let φ be the representation of the group K in the vector space V. We denote by W the subspace of vectors of V which are highest weight vectors of the subalgebra $\mathfrak{m}$: $W = \{v \in V : \mathfrak{m}^+ v = 0\}$.

Let λ be any highest weight of the algebra $\mathfrak{g}$. We denote by $W(\lambda)$ the space of those vectors $w \in W$ for which the following condition hold: for any simple root α, there exists a $\mathfrak{k}_\alpha$-monomorphism

$$U(\mathfrak{k}_\alpha)w \to U(\mathfrak{k}_\alpha)v_\lambda = U(\mathfrak{g}_\alpha)v_\lambda = V_{\lambda_\alpha}, \text{ where } \lambda_\alpha = \lambda|_{\mathfrak{h}_\alpha}. \tag{2}$$

Main Theorem. *The mapping* $\pi \to \pi v_\lambda$ *defines an isomorphism of vector spaces*

$$\mathrm{Hom}_{\mathfrak{k}}(V_\lambda, V) \simeq W(\lambda). \tag{3}$$

§5. Suppose a vector $w \in W$ has the following property: for each simple root $\alpha \in \Pi$, there exists a weight λ_α of the subalgebra $\mathfrak{g}_\alpha$ such that there exists a $\mathfrak{k}_\alpha$-monomorphism $U(\mathfrak{k}_\alpha)w \to V_{\lambda_\alpha}$.

Since $\dim(\mathfrak{h}_\alpha) = 1$, it follows that for every fixed α the weights λ_α are linearly ordered; we denote by $\lambda_\alpha(w)$ the smallest of these.

There exists a unique highest weight $\Lambda = \Lambda(w)$ *of the algebra* $\mathfrak{g}$ *such that for all* α*:* $\Lambda(w)|_{\mathfrak{h}_\alpha} = \lambda_\alpha(w)$.

We denote by Ind φ the representation of G induced by φ, and $I(\varphi)$ denoted the spectrum of Ind φ.

Theorem of the spectrum. *In the space* W *there exists a basis* $\{w_1, \dots, w_n\}$ *such that* $I(\varphi)$ *is the union of* n *"planes", parallel to the spectrum* $I(0)$ *of the representation in the functions on* G/K*:*

$$I(\varphi) = \bigcup_{i=1}^{n} \{\Lambda(w_i) + \lambda,\ \lambda \in I(0)\};$$

$$\text{remind that } I(0) = \left\{ \sum_{i=1}^{p} a_i\lambda_i,\ a_i \in \mathbb{Z},\ a_i \geqslant 0 \right\}. \tag{4}$$

The author wishes to express his gratitude to A. L. Oniščik for suggesting the problem, and for his attention and a number of valuable suggestions.

Part B. Some landmarks in the history of spherical spaces.

In 1975 (DAN SSSR, 220:5, 220:6), due to suggestions of A. L. Oniščik, Yu. V. Dzyadyk found two criteria of sphericity. They were the first results about spherical spaces.

"The results indicated above can be partially extended to some cases which are more general than those we have considered. Indeed, it is not hard to see that they all depend on the lemma on the highest weight vector (§2) and on the theorem on the subalgebras $\mathfrak{g}_\alpha$ (§3). The lemma on the highest weight vector uses only the fact that the algebra $\mathfrak{g}$ has an Iwasawa decomposition" $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} = \mathfrak{k} + \mathfrak{b}$, where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$.

"Let G be a compact Lie group and K a closed subgroup. We will call the homogeneous space G/K *quasi-symmetric* if it admits the following analogue of the Iwasawa decomposition: we can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and an ordering of the root system in such a way that every vector $x \in \mathfrak{g}_+$ can be written as $y + z$, for some $y \in \mathfrak{k}$ and $z \in \mathfrak{h} \oplus \mathfrak{g}_-$. It is clear that every symmetric space is quasi-symmetric.

Proposition. *Let* K/M *be the orbit of general position of the subgroup* K *in a neighborhood of the pole* $o = \{K\} \in G/K$*. Then if the homogeneous space* G/K *is quasi-symmetric, we have the equation* $\dim G/K - \dim K/M = \mathrm{rank}\ G - \mathrm{rank}\ M$".

Part C. Some addenda to the theory of spherical spaces.

§1. The first aim of this part is to close two gaps in the classification of spherical spaces. We describe two classes of spherical spaces, which are absent in the known classifications made by M. Krämer and I. Mykytiuk & M. Brion.

The class of locally symmetric non-symmetric (or almost symmetric) spaces, which are analogous to $\mathrm{SO}_8/\mathrm{Spin}_7$, was completely described in Ukr. Math. J., **50** : 11 (1998).

Now we describe spherical spaces G/K when G is not semisimple.

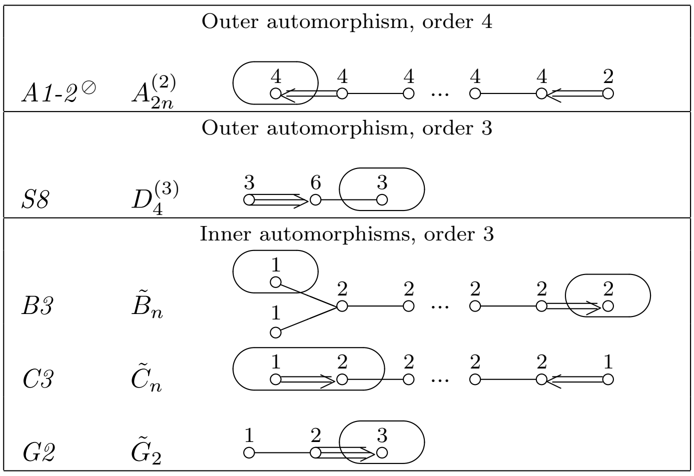
Theorem. *Let* G/K *is simply connected irreducible spherical space, and group* G *is connected reductive non-semisimple. Then* $\dim Z_G = \dim Z_K = 1$*, the space* $(G/Z_G)/K$ *also is*

spherical and belongs to one of the 12 types:
- *symmetric Hermitian – AIII, BDI* (q=2)*, DIII, CI, EIII, EVII;*
- *non–symmetric – A1-2, B3, C3, S10, AA, AC, where A1-2* = $\mathrm{SU}_{2n+1}/\mathrm{Sp}_n\mathrm{U}_1$, *B3* = $\mathrm{SO}_{2n+1}/\mathrm{U}_n$, *C3* = $\mathrm{Sp}_n/\mathrm{Sp}_{n-1}\mathrm{U}_1$, *S10* = $\mathrm{SO}_{10}/\mathrm{Spin}_7\mathrm{SO}_2$, $AA = \mathrm{SU}_n\mathrm{SU}_{n-1}/\mathrm{U}_{n-1}$, $AC = \mathrm{SU}_n\mathrm{Sp}_m/\mathrm{U}_{n-2}\mathrm{Sp}_1\mathrm{Sp}_{m-1}$.

So, we have obtained an exhaustive list of all spherical spaces.

§2. The second aim is to find any system in this classification.

Let us set a problem: what irreducible simply connected spherical spaces (but not symmetric) are determined by automorphism of group G? (It is clear, that group G is simple).
It appeared, of a list of 12 classes irreducible simply connected spherical non–symmetric spaces with simple group G only 5 have this property: *A1-2*$^{\oslash}$= $\mathrm{SU}_{2n+1}/\mathrm{Sp}_n$, *B3, C3, G2* = G_2/A_2, *S8* = Spin_8/G_2. In the next table their Kac schemes are showed:



The next 4 classes are related to symmetric by module of one-dimensional center Z_K: they are *A1-2* = $\mathrm{SU}_{2n+1}/\mathrm{Sp}_n\mathrm{U}_1$, *AIII*$^{\oslash}$($n > 2p > 0$), *DIII*$^{\oslash}$($n = 2k+1 > 1$), *EIII*$^{\oslash}$.
At last, residuary 3 spinor pairs have not any distinguishing automorphism of group G for subgroup K: *S7* = Spin_7/G_2, *S9* = $\mathrm{Spin}_9/\mathrm{Spin}_7$, *S10* = $\mathrm{SO}_{10}/\mathrm{Spin}_7\mathrm{SO}_2$.

Part D. Some results and problems about induced representations over spherical spaces.