V. M. Gichev
Sobolev Institute of Mathematics, Omsk Branch
Omsk, Russia
gichev@iitam.omsk.net.ru

Invariant function algebras on homogeneous spaces

Invariant function algebras on a homogeneous space contain many information (particularly, of the complex geometric nature) about it. Let G be a Lie group, which acts transitively on a manifold M. Suppose that G is compact. The space $C(M)$ of all continuous complex functions on M endowed with the sup-norm is a commutative Banach algebra. The group G acts on $C(M)$ by translations. We say that A is an invariant function algebra on M if it is a closed G-invariant subalgebra of $C(M)$ that contains the constant functions.

The maximal ideal space $\mathcal{M}_A = \mathrm{Hom}(A, \mathbb{C})$ is a geometric object, which keeps the most essential information on A. For example, if $\mathcal{M}_A = M$, then $A = C(M)$ (M is naturally embedded to $\mathcal{M}_A$ as its Shilov boundary). If $G \subset \mathrm{GL}(V)$, where V is a finite dimensional complex linear space, and $M = Gv$, $v \in V$, is an orbit, then $\mathcal{M}_A$ may be identified with the polynomial hull of M. For any compact set $Q \subset V$, its polynomial hull $\widehat{Q}$ is defined as

$$\widehat{Q} = \{z \in V : |p(z)| \leq \sup_{\zeta \in Q} |p(\zeta)| \quad \text{for all} \quad p \in \mathcal{P}(V)\}.$$

If $\widehat{Q} = Q$, then Q is called polynomially convex. The hull of a "generic" Q may be very irregular. The problem of determination of the hulls is certainly insoluble in general. For the orbits Gv, this problem seems to be difficult but soluble. The answer is known if G is the isotropy group of a bounded symmetric domain (Kaup and Zaitzev, 2003; Kaup, 2004). Then the problem can be reduced to the case of the group $S_n\mathbb{T}^n$ acting naturally in $\mathbb{C}^n$, where S_n is the group of all permutations of the coordinates. The hull of the group G in $\mathrm{L}(V)$ is a semigroup; it is determined by the hull of its maximal torus T: $\widehat{G} = G\widehat{T}G$. The polynomially convex orbits are exactly the real forms of closed orbits of $G^{\mathbb{C}}$ (Gichev and Latypov, 2001). A homogeneous space admits an invariant function algebra which is not self-conjugated with respect to the complex conjugation if and only if the isotropy representation has no trivial component (Latypov, 1999; this generalizes results of Gangolli, de Leew, and Wolf of 60s).

Any invariant function algebra A contains the unique maximal invariant ideal J, which is necessarily closed. Adding the constant functions to J, we get an algebra B with the invariant maximal ideal of codimension one (equivalently, G has a fixed point in $\mathcal{M}_B$); factorizing by J, we get an algebra without proper invariant ideals (the norm in A/J is not the sup-norm in general). Thus, one has to consider these opposite cases before the general one. Any invariant function algebra without proper invariant ideals can be realized as the closure of $\mathcal{P}(V)$ in $C(M)$, where $M = Gv$ and $G^{\mathbb{C}}v$ is closed. The algebra is the closure of the set of all smooth CR-functions on M; thus, it is completely determined by the CR-structure and the inner complex geometry of $G^{\mathbb{C}}v$. This also uniquely defines an equivariant embedding of the homogeneous CR-manifold M and its hull $\widehat{M}$ to the complex manifold $G^{\mathbb{C}}v$. The complexified flag manifolds (the closed adjoint orbits in the complex semisimple Lie algebras) is the simplest case. The hulls can be described if the flag manifold is a compact hermitian symmetric space (using the results on the bounded symmetric domains). For the full flags in $\mathbb{C}^n$ the answer is not known yet, even for $n = 3$.

We say that A is finitely generated if it is generated as a Banach algebra by its finite dimensional invariant subspace. Then M, A, and $\mathcal{M}_A = \widehat{Gv}$ can be realized as above in some finite dimensional space V. The invariant function algebras which are not finitely generated have many new properties. For example, if A is finitely generated and G has a fixed point in $\mathcal{M}_A$, then we may assume without loss of generality that $0 \in \widehat{Gv}$. Due to the Hilbert–Mumford criterion, there exists $\xi \in \mathfrak{g}$, such that $\lim_{t\to+\infty} e^{it\xi}v = 0$, where $\mathfrak{g}$ is the Lie algebra of G. In general, this is not true but some analog holds with a chain of one parameter semigroups instead of the single semigroup $e^{it\xi}$.

The case of noncompact G is much more complicated but some of the results and constructions of the compact case also hold for it. For example, the maximal ideal spaces of bi-invariant algebras on Lie groups, under some analytic restrictions, has a natural semigroup structure.