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On rationality of W-algebras

A vertex algebra V, used to construct a rational conformal field theory, must satisfy at least the following three conditions:

(a) V has only finitely many irreducible representations $\{M_j\}_{j\in J}$,

(b) the normalized characters $\chi_j(\tau) = \mathrm{tr}_{\,\mathrm{M_j}} \mathrm{e}^{2\pi \mathrm{i}\tau(\mathrm{L}_0 - c/24)}$ converge to holomorphic functions on the complex upper half-plane $\mathbb{C}^+$,

(c) the functions $\{\chi_j(\tau)\}_{j\in J}$ span an $SL_2(\mathbb{Z})$-invariant space.

A vertex algebra V is called rational if it satisfies these three properties. Recall that any semisimple vertex operator algebra, i.e. a vertex operator algebra for which any representation is completely reducible, is rational, however it is unclear how to verify the semisimplicity condition for vertex algebras considered.

It is well known that lattice vertex algebras, associated to even positive definite lattices, are rational (and semisimple as well). A simple Virasoro vertex algebra is rational (and semisimple as well) iff its central charge is of the form $c = 1 - \frac{6(p-p')^2}{pp'}$, where p, p' are relatively prime integers, greater than 1 (which are central charges of the so called minimal models). A simple affine vertex algebra $V_k(\mathfrak{g})$, attached to a simple Lie algebra $\mathfrak{g}$ is rational (and semisimple as well) iff its level k is a non-negative integer. (Simplicity of a vertex operator algebra is a necessary, but by far not sufficient, condition of semisimplicity.)

It follows from an old paper by Wakimoto and myself that for a rational k of the form

$$k = -h^\vee + \frac{p}{u}, \text{ where } (p,u) = 1\,,\ u \geq 1\,,\ (u,\ell) = 1 \text{ (resp. } = \ell)\,,\ p \geq h^\vee \text{ (resp. } \geq h)\,, \qquad (1)$$

where h is the Coxeter number, $h^\vee$ is the dual Coxeter number and $\ell (= 1, 2 \text{ or } 3)$ is the “lacety” of $\mathfrak{g}$, the normalized character of the vertex algebra $V_k(\mathfrak{g})$ is a modular function (conjecturally, these are all k with this property). We also showed in that for these k with $(u, \ell) = 1$ the affine Lie algebra $\widehat{\mathfrak{g}}$ has a finite set of irreducible highest weight modules $\{M_j\}_{j\in J}$ (called admissible), whose regularized normalized characters $\chi_j(\tau, z) = \mathrm{tr}_{\,\mathrm{M_j}} \mathrm{e}^{2\pi \mathrm{i}\tau(\mathrm{L}_0 - c/24) + \mathrm{z}}$, $\tau \in \mathbb{C}^+$, $z \in \mathfrak{g}$, span an $SL_2(\mathbb{Z})$-invariant space (if $(u, \ell) = \ell$, then one has only $\Gamma_0(\ell)$-invariance). Conjecturally, these $\widehat{\mathfrak{g}}$-modules extend to $V_k(\mathfrak{g})$ and are all of its irreducible modules (this conjecture was proved Adamovic and Milas for $\mathfrak{g} = s\ell_2$). Thus, $V_k(\mathfrak{g})$ for k of the form (1) with $(u, \ell) = 1$ satisfy the properties (a) and (c) of rationality, but property (b) fails for some $j \in J$ since $\chi_j(\tau, z)$ may have a pole at $z = 0$, unless k is a non-negative integer.

In my talk I shall discuss, following a joint work with M. Wakimoto, the problem of rationality of simple W-algebras $W_k(\mathfrak{g}, f)$, a family of vertex algebras, depending on $k \in \mathbb{C}$, attached to a simple Lie algebra $\mathfrak{g}$ and a nilpotent element f of $\mathfrak{g}$ (rather its conjugacy class), which has been intensively studied in the mathematics and physics literature.

More precisely, one needs to analyze for which triples $(\mathfrak{g}, f, k)$ the $W_k(\mathfrak{g}, f)$-modules, obtained by the quantum Hamiltonian reduction from admissible modules of level k over the affine Lie algebra $\widehat{\mathfrak{g}}$, have convergent characters, as the modular invariance property is preserved by this reduction. We call such f an exceptional nilpotent and such k an exceptional level.

The most well studied case of W-algebras is that corresponding to the principal nilpotent element f (a special case of which for $\mathfrak{g} = s\ell_2$ is the Virasoro vertex algebra). It follows from a paper by E. Frenkel, Wakimoto and myself that for principal f the exceptional levels k are given by

$$k = -h^\vee + \frac{p}{u}, \text{ where } (p,u) = 1\,,\ p \geq h^\vee\,,\ u \geq h\,,\ (u,\ell) = 1\,. \qquad (2)$$

It is expected that for the principal nilpotent f, (2) are precisely the values of k, for which $W_k(\mathfrak{g}, f)$ is a semisimple vertex algebra.

Surprisingly, beyond the principal nilpotent, there are very few exceptional nilpotents. We conjecture that there exists an order preserving map of the set of non-principal exceptional nilpotent orbits of $\mathfrak{g}$ to the set of positive integers, relatively prime to ℓ and smaller than h, such that the corresponding integer u is the only denominator of an exceptional level $k = -h^\vee + p/u$, where $p \geq h^\vee$ and $(u, p) = 1$.

Note that $f = 0$ is an exceptional nilpotent, corresponding to $u = 1$, since in this case $W_k(\mathfrak{g}, 0)$ is the simple affine vertex algebra of level $k \in \mathbb{Z}_+$.

We prove the above conjecture for $\mathfrak{g} \simeq s\ell_n$. In this case the above map is bijective to the set $\{1, 2, \ldots, n-1\}$, and the exceptional nilpotent, corresponding to the positive integer $u \leq h = n$, is given by the partition $n = u + \cdots + u + s$, where $0 \leq s < u$.

For an arbitrary simple $\mathfrak{g}$ we give a geometric description of the exceptional pairs (k, f) in terms of $\mathfrak{g}$ and its adjoint group.