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Varieties of dialgebras and conformal algebras

In this talk we present a natural relation between dialgebras (introduced by J.-L. Loday in [L]) and conformal algebras (by V.G. Kac [K]).

By definition, a dialgebra is a linear space with two (non-related, in general) bilinear operations $(\cdot \dashv \cdot)$, $(\cdot \vdash \cdot)$. A dialgebra is said to be associative if it satisfies five identities chosen in such a way that the new operation $[xy] = x \vdash y - y \dashv x$ turns the dialgebra into a left Leibniz algebra. In a similar way, D. Liu [LD] introduced alternative dialgebras in relation with Steinberg–Lie algebras. Dialgebras and Leibniz algebras also give rise to so called perm-algebras [CF] and quasi-Jordan algebras [VF].

Conformal algebras, originated from mathematical physics, are linear spaces (over a field of zero characteristic) endowed with a linear transformation T and with a countable family of bilinear operations $(\cdot\, {}_{(n)}\, \cdot)$, $n \in \mathbb{Z}_+$, satisfying certain axioms coming from the properties of the operator product expansion (OPE) in conformal field theory.

Every conformal algebra C can be canonically embedded into the space of formal power series $A[[z, z^{-1}]]$ over an appropriate algebra A called annihilation (or coefficient) algebra of C. A conformal algebra is said to be associative (Lie, alternative, etc.) iff so is its annihilation algebra.

It turns out that dialgebras and conformal algebras are related in the following way. Given a conformal algebra C one may obtain the dialgebra structure on the same space: operations

$$x \dashv y = x\, {}_{(0)}\, y, \quad x \vdash y = \sum_{s\geq 0} \frac{1}{s!}(-T)^s (x\, {}_{(s)}\, y).$$

The dialgebra obtained is denoted by $C^{(0)}$.

We present a general and natural scheme how to define what is a variety Var of dialgebras corresponding to a given variety of ordinary algebras. The appropriate language for this definition is provided by the notion of an operad. An algebra over a field $\Bbbk$ can be considered as a functor from the operad Alg of binary trees to the multi-category $\mathrm{Vec}_\Bbbk$ of linear spaces with polylinear maps. If we replace $\mathrm{Vec}_\Bbbk$ with the multi-category $\mathcal{M}^*(H)$ of left H-modules, where H is a cocommutative Hopf algebra (see, e.g., [BDK] for the description of $\mathcal{M}^*(H)$), then we obtain the definition of a pseudo-algebra over H. A pseudo-algebra over $H = \Bbbk[T]$ is exactly what is known as a conformal algebra.

Suppose Var is a homogeneous variety of algebras defined by a system of poly-linear identities. Then there exists an operad VarAlg and a full functor $\mathrm{Var} : \mathrm{Alg} \to \mathrm{VarAlg}$ such that an algebra $A : \mathrm{Alg} \to \mathrm{Vec}_\Bbbk$ belongs to Var iff there exists a functor $\bar{A} : \mathrm{VarAlg} \to \mathrm{Vec}_\Bbbk$ such that $A = \mathrm{Var} \circ \bar{A}$, see, e.g., [GK]. Using $\mathcal{M}^*(H)$ instead of $\mathrm{Vec}_\Bbbk$ we obtain what is a pseudo-algebra of the variety Var (note that the class of all these pseudo-algebras does not form a variety in the ordinary sense).

In a similar way, a dialgebra is a functor from the operad Dialg of binary trees with 2-colored vertices (colors 1 and 2 correspond to operations $\vdash$ and $\dashv$) to the multi-category $\mathrm{Vec}_\Bbbk$. Note that all dialgebra structures that appear in the literature satisfy the following identities:

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad x \dashv (y \dashv z) = x \vdash (y \dashv z).$$

These identities define a homogeneous variety of *0-dialgebras.* Let us denote the corresponding operad by Dialg_0.

Theorem 1. *The operad* Dialg_0 *is equivalent to* $\mathrm{Alg} \otimes \mathcal{E}$, *where* $\mathcal{E}$ *is the operad of finite-dimensional vector spaces.*

The last statement is important since it leads to the following natural definition. A dialgebra A is said to be a Var-dialgebra if there exists a functor $\bar{A} : \mathrm{VarAlg} \otimes \mathcal{E} \to \mathrm{Vec}_{\Bbbk}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Dialg} & \xrightarrow{A} & \mathrm{Vec}_{\Bbbk} \\ \downarrow & & \uparrow \bar{A} \\ \mathrm{Dialg}_0 \simeq \mathrm{Alg} \otimes \mathcal{E} & \xrightarrow{\mathrm{Var}\otimes\mathrm{id}} & \mathrm{VarAlg} \otimes \mathcal{E} \end{array}$$

is commutative. In particular, we obtain the structures earlier used in associative and alternative cases; the class of Lie dialgebras coincides with the class of Leibniz algebras; commutative and Jordan dialgebras give rise to perm-algebras and quasi-Jordan algebras, respectively.

Theorem 2. *An arbitrary* Var*-dialgebra can be embedded into an appropriate pseudoalgebra of the variety* Var *over* $H = \Bbbk[T]$.

We introduce the notion of a *conformal representation* of a Leibniz algebra (Lie dialgebra), and prove that a (finite-dimensional) Leibniz algebra has a (finite) faithful conformal representation. This implies various important corollaries for Leibniz algebras and associative dialgebras.

Corollary [L]**.** *The universal enveloping associative dialgebra $U(L)$ of a Leibniz algebra L is isomorphic (as a linear space) to $S(L^{alg}) \otimes L$, where L^{alg} is the maximal Lie image of L.*

Corollary. *A finite-dimensional Leibniz algebra L can be embedded into the matrix algebra $M_n(\Bbbk[T])$ over the polynomial ring with respect to the operation*

$$[a(T)b(T)] = a(0)b(T) - b(T)a(0).$$

In particular, L can be embedded into a finite-dimensional associative dialgebra.

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