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Descent of line bundles to GIT quotients of flag varieties

Let G be a connected semisimple complex algebraic group with a maximal torus T and let P be a parabolic subgroup containg T. We denote their Lie algebras by the corresponding Gothic characters. The following theorem is our main result.

Theorem. Let $\mathcal{L}(\lambda)$ be a homogeneous ample line bundle on the flag variety $X = G/P$. Then, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle on the GIT quotient $X^{ss}(\lambda)//T$ (i.e., there exists a line bundle $\mathcal{L}$ on $X^{ss}(\lambda)//T$ whose pull-back to $X^{ss}(\lambda)$ is the restriction of $\mathcal{L}(\lambda)$) if and only if for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular, rank $\mathfrak{s}$ = rank $\mathfrak{g}$),

$$\lambda \in \sum_{\alpha \in \triangle_+(\mathfrak{s})} \mathbb{Z}\alpha,$$

where $\triangle_+(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

As a consequence of the above theorem, we get precisely which line bundles descend to the geometric quotients $X^{ss}(\lambda)//T$.

In the following Q (resp., Λ) is the root (resp., weight) lattice and we follow the indexing convention as in Bourbaki.

Theorem. Let G be a connected, simply-connected simple algebraic group, $P \subset G$ a parabolic subgroup and let $\mathcal{L}(\lambda)$ be a homogeneous ample line bundle on the flag variety $X = G/P$. Then, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle on the GIT quotient $X^{ss}(\lambda)//T$ if and only if λ is of the following form depending upon the type of G.

a) G of type $A_\ell\, (\ell \geq 1) : \lambda \in Q$

b) G of type $B_\ell\, (\ell \geq 3) : \lambda \in 2Q$

c) G of type $C_\ell\, (\ell \geq 2) : \lambda \in \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$

d1) G of type $D_4 : \lambda \in \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_i : n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even$\}$.

d2) G of type $D_\ell\, (\ell \geq 5) : \lambda \in \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell, n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even $\}$.

e) G of type $G_2 : \lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$.

f) G of type $F_4 : \lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$.

g) G of type $E_6 : \lambda \in 6P$.

h) G of type $E_7 : \lambda \in 12P$

i) G of type $E_8 : \lambda \in 60Q$.