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Is the Luna stratification intrinsic?

This is joint work with Zinovy Reichstein. Let G be a reductive group over an algebraically closed field k of characteristic zero.

If $V = \operatorname{Spec} A$ is an affine G-variety, then V admits a categorical quotient

$$\pi\colon V \to X = V/\!/G = \operatorname{Spec} A^G.$$

X carries a natural stratification due to D. Luna: for each point $x \in X$ let $v_x \in V$ be a point such that Gv_x is the unique closed G-orbit in the fiber $\pi^{-1}(x)$. Then the stabilizer G_{v_x} is a reductive subgroup determined by x up to conjugacy in G. For a reductive subgroup $H \subset G$ let (H) denote its conjugacy class, then $X^{(H)} := \{x \in X \mid G_{v_x} \in (H)\}$ is the stratum associated to H or (H). This is indeed a stratification: each stratum is locally closed, and the closure of each stratum is the disjoint union of strata. There are only finitely many strata. If V is irreducible and non-singular, then each Luna stratum is irreducible and non-singular as well. In the following V will always be a representation of G.

It is a natural question to ask in how far the Luna stratification of X is a property of X as a variety (i.e. forgetting that X is a quotient). One aspect of this question is related to the automorphisms of X:

i) Is the Luna stratification intrinsic? That is, is the stratification preserved by every automorphism σ of X? In other words, is for every stratum $S \subset X$, $\sigma(S)$ again a Luna stratum?

ii) Related but stronger is the question whether individual Luna strata are intrinsic, i.e. is $\sigma(S) = S$ for every automorphism σ and every Luna stratum $S \subset X$?

It is clear that the answer to both questions is no in general. Indeed, if V is a coregular representation of G, then X is isomorphic to some affine space; no finite (non-trivial) stratification of an affine space of positive dimension is intrinsic in the above sense, as we can always map any two points in a stratum S, say, to two points in different strata.

Let V^r denote the direct sum of r copies of V. It is a somewhat vague principle in Invariant Theory that the action of G on V^r sometimes "improves" as r increases, and hence the quotients $V^r/\!/G$ are better understood. Our first result is a manifestation of this: If W is a representation of G and $V = W^r$ then the Luna stratification of $V/\!/G$ is intrinsic whenever $r \geq 2 \dim W$. If W is orthogonal, then $r \geq \dim W + 1$ suffices. Finally, if $W = \operatorname{Lie}(G)$ is the adjoint representation, then the stratification of V is intrinsic whenever $r \geq 3$.

In all cases the underlying reason is the fact that the closure of each Luna stratum S is singular at each point away from the open stratum S.

The starting point of our investigation was actually the adjoint representation in the special case of GL_n acting on the $n \times n$-matrices M_n by conjugation. Earlier, Reichstein had constructed automorphisms of $M_n^r/\!/\mathrm{GL}_n$ ($r \geq n+1$) which all preserve the stratification, and in fact preserve individual strata as well. He conjectured this to be true for all automorphisms of $M_n^r/\!/\mathrm{GL}_n$. We prove this conjecture whenever $r \geq 3$.