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Uniqueness properties for spherical varieties

The base field $\mathbb{K}$ is assumed to be algebraically closed and of characteristic zero. Let G be a connected reductive group. Fix a Borel subgroup $B \subset G$. An irreducible G-variety X is said to be *spherical* if X is normal and B has an open orbit on X. The last condition is equivalent to $\mathbb{K}(X)^B = \mathbb{K}$. Note that a spherical G-variety contains an open G-orbit.

The goal of this note is to review the author results on uniqueness properties for certain classes of spherical varieties: homogeneous spaces, smooth affine varieties and general affine varieties. The properties are stated in terms of certain combinatorial invariants.

Let us describe combinatorial invariants in interest. Fix a maximal torus $T \subset B$.

The set $\mathfrak{X}_{G,X} := \{\mu \in \mathfrak{X}(T) | \mathbb{K}(X)^{(B)}_\mu \neq \{0\}\}$ is called the *weight lattice* of X. This is a sublattice in $\mathfrak{X}(T)$. By the *Cartan space* of X we mean $\mathfrak{a}_{G,X} := \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{Q}$. This is a subspace in $\mathfrak{t}(\mathbb{Q})^*$.

Next we define the valuation cone of X. Let v be a $\mathbb{Q}$-valued discrete G-invariant valuation of $\mathbb{K}(X)$. One defines the element $\varphi_v \in \mathfrak{a}^*_{G,X}$ by the formula

$$\langle \varphi_v, \mu \rangle = v(f_\mu), \forall \mu \in \mathfrak{X}_{G,X}, f_\mu \in \mathbb{K}(X)^{(B)}_\mu \setminus \{0\}.$$

It is known that the map $v \mapsto \varphi_v$ is injective. Its image is a finitely generated convex cone in $\mathfrak{a}^*_{G,X}$. We denote this cone by $\mathcal{V}_{G,X}$ and call it the *valuation cone* of X.

Let $\mathcal{D}_{G,X}$ denote the set of all prime B-stable divisors of X. This is a finite set. To $D \in \mathcal{D}_{G,X}$ we assign $\varphi_D \in \mathfrak{a}^*_{G,X}$ by $\langle \varphi_D, \mu \rangle = \mathrm{ord}_D(f_\mu), \mu \in \mathfrak{X}_{G,X}, f_\mu \in \mathbb{K}(X)^{(B)}_\mu \setminus \{0\}$. Further, for $D \in D_{G,X}$ set $G_D := \{g \in G | gD = D\}$. Clearly, G_D is a parabolic subgroup of G containing B. Choose $\alpha \in \Pi(\mathfrak{g})$. Below we regard $\mathcal{D}_{G,X}$ as an abstract set equipped with two maps $D \mapsto \varphi_D, D \mapsto G_D$.

Theorem 1 ([2]). *Let X_1, X_2 be spherical homogeneous spaces of G. If $\mathfrak{X}_{G,X_1} = \mathfrak{X}_{G,X_2}, \mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}, \mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2}$, then X_1, X_2 are equivariantly isomorphic.*

Now we consider uniqueness properties for affine spherical varieties. A basic combinatorial invariant of an affine spherical G-variety X is its weight monoid $\mathfrak{X}^+_{G,X} := \{\lambda | f_\lambda \in \mathbb{K}[X]\}$. It is clear that

$$\mathfrak{X}^+_{G,X} = \{\lambda \in \mathfrak{X}_{G,X} | \langle \varphi_D, \lambda \rangle \geqslant 0, \forall D \in \mathcal{D}_{G,X}\}.$$

The next theorem incorporates uniqueness properties for both smooth and arbitrary affine spherical varieties. It is proved using Theorem 1.

Theorem 2 ([1]). *Let X_1, X_2 be affine spherical G-varieties such that $\mathfrak{X}^+_{G,X_1} = \mathfrak{X}^+_{G,X_2}$. Suppose at least one of the following conditions holds:*

1. *Both X_1, X_2 are smooth.*
2. *$\mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}$.*

Then X_1, X_2 are G-equivariantly isomorphic.

References

[1] I.V. Losev. *Proof of the Knop conjecture.* Preprint (2006), arXiv:math.AG/0612561v4, 20 pages.
[2] I.V. Losev. *Uniqueness property for spherical homogeneous spaces.* Preprint (2007), arXiv:math.AG/0703543, 21 pages.