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Polynomial quantization on para-Hermitian symmetric spaces

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We construct a variant of quantization (symbol calculus) in the spirit of Berezin on para-Hermitian symmetric spaces. A general scheme of quantization was presented in [1]. There are 4 classes of symplectic semisimple symmetric spaces G/H: (a) Hermitian symmetric spaces; (b) semi-Kählerian symmetric spaces; (c) para-Hermitian symmetric spaces; (d) complexifications of Hermitian symmetric spaces. Spaces of class (a) are Riemannian, spaces of other three classes are pseudo-Riemannian (non-Riemannian). Let us assume that G is a *simple* Lie group. Then these 4 classes give a classification.

Berezin constructed quantization for spaces of class (a). We consider spaces of class (c). We can assume that G/H is an adjoint G-orbit. The Lie algebra $\mathfrak{g}$ of G splits into the direct orthogonal (in sense of the Killing form) sum: $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ where $\mathfrak{h}$ is the Lie algebra of H. The space $\mathfrak{q}$ splits into the direct sum of Lagrangian subspaces $\mathfrak{q}^-$ and $\mathfrak{q}^+$ of the tangent space to G at the initial point H. The subspaces $\mathfrak{q}^\pm$ are invariant and irreducible with respect to H, they are Abelian subalgebras of $\mathfrak{g}$. The pair $(\mathfrak{q}^+, \mathfrak{q}^-)$ is a Jordan pair. Let r and $\varkappa$ be rank and genus of it, r being also rank of G/H.

Set $Q^\pm = \exp \mathfrak{q}^\pm$. The subgroups $P^\pm = HQ^\pm = Q^\pm H$ are maximal parabolic subgroups of G. We have the following decompositions (the Gauss and "anti-Gauss" decompositions): $G = \overline{Q^+HQ^-}$, $G = \overline{Q^-HQ^+}$, where the bar means closure. The group G acts on $\mathfrak{q}^-$ and $\mathfrak{q}^+$: $\xi \mapsto \widetilde{\xi}$, $\eta \mapsto \widehat{\eta}$, where $\widetilde{\xi}$ and $\widehat{\eta}$ are taken from the Gauss and the anti-Gauss decompositions:

$$\exp \xi \cdot g = \exp Y \cdot \widetilde{h} \cdot \exp \widetilde{\xi}, \quad \exp \eta \cdot g = \exp X \cdot \widehat{h} \cdot \exp \widehat{\eta}, \tag{1}$$

Therefore, G acts on $\mathfrak{q}^- \times \mathfrak{q}$, the stabilizer of the point $(0,0)$ is H, so that we obtain an embedding $\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H$ with an open and dense image. Let us call ξ, η *horospherical coordinates* on G/H. For $\xi \in \mathfrak{q}^-$ and $\eta \in \mathfrak{q}^+$, let us decompose the anti-Gauss product $\exp \xi \cdot \exp(-\eta)$ according to the Gauss decomposition and denote by $h(\xi, \eta)$ the corresponding element in H. For $h \in H$, let us denote $b(h) = \det(\mathrm{Ad} h)|_{\mathfrak{q}^+}$. The function $k(\xi, \eta) = b(h(\xi, \eta))$ is an analogue of the Bergman kernel for Hermitian symmetric spaces. It is $N(\xi, \eta)^{-\varkappa}$, where $N(\xi, \eta)$ is an irreducible polynomial in ξ and η of degree r in ξ and η separately.

Representations $\pi_\lambda^\pm$, $\lambda \in \mathbb{C}$, of G of a maximal degenerate series associated with G/H are defined as induced representations $\pi_\lambda^\pm = \mathrm{Ind}(G, P^\mp, \omega_{\mp\lambda})$, where $\omega_\lambda(h) = |b(h)|^{-\lambda/\varkappa}$ and $\omega_\lambda = 1$ on $Q^\pm$. In noncompact picture, these representations act on functions $\varphi(\xi)$ and $\psi(\eta)$ on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively by (see (1)):

$$(\pi_\lambda^-(g)\varphi)(\xi) = \omega_\lambda(\widetilde{h})\varphi(\widetilde{\xi}), \quad (\pi_\lambda^+(g)\psi)(\eta) = \omega_\lambda(\widehat{h})\psi(\widehat{\eta}).$$

An operator $A_{-\lambda-\varkappa}$ with the kernel $\Phi_\lambda(\xi, \eta) = |N(\xi, \eta)|^\lambda$ intertwines $\pi^\pm_{-\lambda-\varkappa}$ with $\pi_\lambda^\mp$. The product $A_\lambda A_{-\lambda-\varkappa}$ is $c(\lambda)^{-1} \cdot \mathrm{id}$, where $c(\lambda)$ is a meromorphic function of λ.

For the initial algebra of operators, we take the algebra of operators $D = \pi_\lambda^-(X)$, where X belongs to the universal enveloping algebra $\mathrm{Env}(\mathfrak{g})$ of $\mathfrak{g}$. This algebra acts on functions $\varphi(\xi)$ and $\psi(\eta)$ by representations π_λ^- and π_λ^+ respectively. Spaces of these functions form analogues of the Fock space. For the supercomplete system we take the kernel $\Phi_\lambda(\xi, \eta)$. Let us call the *covariant symbol* of the operator $D = \pi_\lambda^-(X)$, $X \in \mathrm{Env}\,(\mathfrak{g})$, the following function F on G/H which in horosherical coordinates is given by

$$F(\xi, \eta) = \Phi_\lambda(\xi, \eta)^{-1}(\pi_\lambda^-(X) \otimes 1)\Phi_\lambda(\xi, \eta).$$

These functions are polynomials on G/H. It is why we call our version quantization the *polynomial quantization*. For λ generic, the space of covariant symbols is the space of all polynomials on G/H. The operator D is recovered by its covariant symbol F as follows:

$$(D\varphi)(\xi) = c(\lambda) \int F(\xi, v) \, \frac{\Phi_\lambda(\xi, v)}{\Phi_\lambda(u, v)} \, \varphi(u) \, dx(u, v), \tag{2}$$

where $dx(\xi, \eta)$ is a G-invariant measure on G/H. The correspondence $D \mapsto F$ is $\mathfrak{g}$-equivariant. The multiplication of operators gives raise to a multiplication $\star$ of covariant symbols. It is given by the *Berezin kernel* $\mathcal{B}_\lambda$:

$$(F_1 * F_2)(\xi, \eta) = \int F_1(\xi, v) F_2(u, \eta) \mathcal{B}_\lambda(\xi, \eta; u, v) \, dx(u, v),$$

where

$$\mathcal{B}_\lambda(\xi, \eta; u, v) = c(\lambda) \, \frac{\Phi_\lambda(\xi, v)\Phi_\lambda(u, \eta)}{\Phi_\lambda(\xi, \eta)\Phi_\lambda(u, v)}.$$

A function (a polynomial) $F(\xi, \eta)$ is the *contravariant* symbol for an operator A such that $(A\varphi)(\xi)$ is given by the right hand side of (2) with $F(\xi, v)$ replaced by $F(u, v)$.

Thus, we have two maps: $\mathcal{O}_\lambda = (\text{contra}) \circ (\text{co})$ and the *Berezin transform* $\mathcal{B}_\lambda = (\text{co}) \circ (\text{contra})$. The map $\mathcal{O}_\lambda$ (it was absent in Berezin's theory) assigns to an operator D with the covariant symbol F the operator A for which F is the contravariant symbol. The kernel of A is obtained from the kernel of D by the permutation of arguments and replacing λ by $-\lambda - \varkappa$. The map $\mathcal{B}_\lambda$ assigns to the contravarint symbol F of an operator D the covariant symbol F of the same D. It is given just by the Berezin kernel.

Let us formulate open problems for arbitrary rank r: find an expression of the Berezin transform $\mathcal{B}_\lambda$ in terms of Laplace operators, find eigenvalues of $\mathcal{B}_\lambda$ on irreducible constituents, find its full asymptotic expansion of $\mathcal{B}_\lambda$ when $\lambda \to -\infty$. These problems are solved for $r = 1$, see [2], and for spaces with $G = \mathrm{SO}_0(p, q)$ (then $r = 2$).

There is another approach to the polynomial quantization using representation theory. It gives co- and contravariant symbols and the Berezin transform in a natural and transparent way. These symbols are obtained under the restriction of a representation R_λ of the overgroup $\widetilde{G} = G \times G$ to the component subgroups $G \times e$ and $e \times G$ ($R_\lambda(g_1, g_2) = \pi_\lambda^-(g_2) \otimes \pi_\lambda^+(g_1)$).

References

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