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Degree bounds, the regular representation, and invariants of permutation groups in general

Let $\mathbb{F}$ be a field, G a finite group, and consider a faithful representation of G

$$\rho : G \hookrightarrow \mathsf{GL}(n, \mathbb{F})$$

of degree n over $\mathbb{F}$. Via ρ the group G acts on the vector space $V = \mathbb{F}^n$, hence on the dual space V^*, and thus on the symmetric algebra on the dual, denoted by $\mathbb{F}[V]$. We are interested in the subalgebra

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] | gf = f \ \forall g \in G\} \hookrightarrow \mathbb{F}[V]$$

of polynomial invariants of G. In particular, we want to focus on permutation representations, i.e.,

$$\rho(G) \leq \mathsf{GL}(n,\ \mathbb{F})$$

consists of permutation matrices.

We note that since every field has a 0 and a 1, permutation representations live over every field. Thus is makes sense to compare rings of invariants of the same permutation groups over different fields.

If $\mathbb{F} \geq \mathbb{K}$ is a field extension then

$$\mathbb{F}[V]^G = \mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}[V]^G$$

i.e., this case is not interesting. Instead, we want to compare rings of invariants of permutation groups across characteristics.

There are several classical results about invariants of permutation representation that are valid over any ground field $\mathbb{F}$:

The basic example is probably given by the invariants of the symmetric group in its defining representation. They are, of course, generated by the elementary symmetric functions *over any ground field*. More generally for any field $\mathbb{F}$ we have

- $\mathbb{F}[V]^G$ is generated as an $\mathbb{F}$-algebra by orbit sums of monomials.
- $\mathbb{F}[V]^G$ is generated by orbit sums of degree at most $\min\{n, \binom{n}{2}\}$. (Goebel 1996)
- The Poincaré series of a ring of invariants of a permutation group is independent of the ground field.

If we want further results we discover that the classical invariant theoretic/representation theoretic dichotomy nonmodular-modular is not really relevant, but rather the dichotomy Cohen-Macaulay vs. Non-Cohen-Macaulay invariants. The fundamental result which justifies such a statement is the following:

Theorem 1 (MDN 2007). *Let $\mathbb{F}$ and $\mathbb{K}$ be fields. Assume that $\mathbb{F}[V]^G$ and $\mathbb{K}[V]^G$ are both Cohen-Macaulay. Then there is a set $\{f_1, \ldots, f_k\}$ of orbit sums of monomials that is a minimal algebra generating set for both rings simultaneously.*

It is this result which we want to explain and put into context during the talk.

Since it allows us to compare Cohen-Macaulay permutation invariants *across characteristics*, in particular we can compare them with an invariant ring with a characteristic zero ground field. It thus leads to many interesting corollaries on Cohen-Macaulay invariant rings of permutation representations, like KilliusĆonjecture, the relative Noether bound, or the fact that a Cohen-Macaulay ring of permutation invariants is polynomial if and only if the group is generated by pseudoreflections. We want to present these and several more results, that show that rings of permutation invariants "look" like characteristic zero rings of invariants as long as they are Cohen-Macaulay.

Moreover, the above mentioned theorem allows us also to prove an old conjecture on rings of invariants of *arbitrary* representations:

Since rings of polynomial invariants of finite groups are finitely generated graded $\mathbb{F}$-algebras, we define $\beta(\mathbb{F}[V]^G)$ to be the maximal degree of an $\mathbb{F}$-algebra generator in a minimal algebra generating set. An on-going theme in invariant theory of finite groups is the problem of finding upper bounds on $\beta(\mathbb{F}[V]^G)$. For example, in the nonmodular case it is known that

$$\beta(\mathbb{F}[V]^G) \leq |G|$$

independent of the representation (Noether 1916, Fleischmann 2000, Fogarty 2001). There are many refinements (for special classes of groups or representations) as well as relative versions of this result. Moreoever, in 1989 Schmid fixed a group G and compared degree bounds across representations in characteristic zero and found

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[\mathbb{F}G]^G),$$

where $\mathbb{F}G$ denotes the regular representation. Schmidś degree bound is not valid in the modular case as the threefold regular representation of $\mathbb{Z}/2$ over a field of characteristic 2 shows:

$$\beta(\mathbb{F}_2[(\mathbb{F}_2\mathbb{Z}/2)^{\oplus 3}]^{\mathbb{Z}/2}) = 3 > 2 = \beta(\mathbb{F}_2[\mathbb{F}_2\mathbb{Z}/2]^{\mathbb{Z}/2}).$$

However, it has been conjectured since that Schmidś bound remains valid in the general nonmodular case. Indeed, it had been proven for $char(\mathbb{F}) = p > |G|$ by Smith (2001), for nonmodular representations of *abelian* groups by Sezer (2002), and for nonmodular representations with $char(\mathbb{F}) = p > \frac{3}{8}|G| + 1$ by Knop (2004). Theorem 1 above allows us to prove Schmidś degree bound in the general nonmodular case thus confirming that the regular representation is the "worst" case:

Theorem 2 (MDN 2007). *Let $\rho : G \hookrightarrow \mathsf{GL}(n, \mathbb{F})$ be an arbitrary nonmodular representation. Then*

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[\mathbb{F}G]^G).$$