G. I. Olshanski
Dobrushin Math Lab, Institute for Information Transmission Problems
Moscow, Russia
olsh@online.ru

Thoma's theorem, quasi–symmetric functions, and the boundary of the graph of zigzag diagrams

I will talk about a joint work with Alexander Gnedin. It is published in Internat. Math. Research Notices 2006, Article ID 51968; arXiv:math/0508131. The problem that we solved is related to Thoma's theorem (1964) concerning representations of the infinite symmetric group S_∞.

Thoma discovered that the group S_∞ possesses a large family of "characters" (provided this notion is properly understood) and, moreover, the characters can be completely classified and explicitly constructed. In particular, they are parametrized, in a natural way, by the points of an infinite–dimensional compact space Ω called Thoma's simplex.

This remarkable result can be viewed as an "infinite–dimensional" analog of classical Frobenius' description of irreducible characters of the finite symmetric groups (1900). Vershik and Kerov (1981) found an alternative proof of Thoma's theorem which explains how the points of Thoma's simplex arise from Young diagrams.

Vershik and Kerov also observed that the characters of S_∞ are in a 1–1 correspondence with normalized nonnegative multiplicative functionals on the algebra Λ of symmetric functions over the reals. Here the nonnegativity condition means that a functional takes nonnegative values on the distinguished basis in Λ formed by the Schur functions, and normalization means that the value on the (unique) Schur function of degree 1 equals 1.

This combinatorial interpretation is very important as it opens the way to various generalizations.

One possibility was examined by Kerov, Okounkov, and Olshanski (1998): In their work the Schur functions were replaced by the Jack symmetric functions which form a one–parameter deformation of the Schur functions. This can be compared to Heckman–Opdam's philosophy leading to a generalization of the theory of spherical functions on symmetric spaces.

In our work with Gnedin we explore a different possibility: Specifically, we take instead of the algebra of symmetric functions Λ the larger algebra of quasi–symmetric functions $QSym$. The algebra $QSym$ was introduced by Gessel (1984) and was extensively studied by combinatorialists. Similarly to Λ, $QSym$ is a Hopf algebra, only its comultiplication is noncommutative. The dual Hopf algebra is the algebra $NSym$ of noncommutative symmetric functions of Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon (1995). The algebra $QSym$ possesses a distinguished basis of the so–called fundamental quasi–symmetric functions, which play the role of the Schur functions and which we use to define nonnegative functionals on $QSym$.

Our main result is an explicit description of normalized nonnegative multiplicative functionals on the algebra $QSym$. We show, in particular, that the corresponding space of parameters is a kind of fibration over the Thoma simplex.

As shown by Vershik and Kerov, one more face of Thoma's theorem is that it describes the "boundary" of the Young graph, an infinite graph whose vertices are Young diagrams. Our result also admits a similar interpretation, with a different graph of combinatorial origin whose vertices are the so–called zigzag diagrams introduced by MacMahon.