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On homogeneous supermanifolds associated with irreducible compact Hermitian symmetric spaces

We consider the following problem: given an irreducible compact Hermitian symmetric space $M = G/P$, where G is a simply connected simple complex Lie group and P its parabolic subgroup, to describe, up to isomorphy, all homogeneous complex supermanifolds $(M, \mathcal{O})$, whose reduction is M. An obvious example is the split supermanifold (M, Ω), where Ω is the sheaf of holomorphic differential forms on M. It was proved in [1] that the only non-split homogeneous supermanifolds with this retract are the so-called Π-symmetric super-Grassmannians $\Pi Gr_{n|n,k|k}$, constructed in [2]; in this case $M = \mathrm{G}r_{n,k},\ 0 < k < n$, is the Grassmannian of k-subspaces in $\mathbb{C}^n$. Several results concerning classification of non-split homogeneous supermanifolds $(M, \mathcal{O})$ are known, where the retract is not fixed, but the odd part m of the dimension $n|m = \dim(M, \mathcal{O})$ does not exceed a given number, and also in the case $M = \mathbb{CP}^1$. Here we consider the classification problem under certain restrictions on the representation φ of the subgroup P which determines the retract of $(M, \mathcal{O})$. Another interpretation of φ is that it is dual to the "odd isotropy representation" of P acting in the odd tangent space to $(M, \mathcal{O})$ at the point which is fixed under P. In the split case, we solve the problem for all completely reducible representations φ, while in the non-split case the very strong assumption that φ is irreducible is made. We prove that if this condition is satisfied, then in certain cases φ is dual to the usual isotropy representation, i.e., the retract is isomorphic to (M, Ω). These cases are as follows: $M = \mathrm{Gr}_{n,2}$, where $n \geq 5$ is odd or $n = 4, 6$; $M = \mathrm{Gr}_{n,k},\ 3 \leq k \leq n-k$; $G = Sp_{2n}(\mathbb{C}),\ n \geq 2$ (M is the symplectic isotropic Grassmannian); $G = E_6$; $G = E_7$. Thus, for the Grassmannians listed above the only solution is $\Pi\mathrm{Gr}_{n|n,k|k}$, while in the remaining cases no non-split homogeneous supermanifolds exist. For other irreducible compact Hermitian symmetric spaces M we obtain a list of possible irreducible representations φ, but we could not find any example of non-split homogeneous supermanifolds with reduction M and an irreducible odd isotropy representation except of $\Pi\mathrm{Gr}_{n|n,k|k}$. The proofs are published in [3, 4].

References

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