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Invariants and orbits of the triangular group

Let $N = \mathrm{UT}(n, K)$ be the subgroup of lower triangular matrices of size n with units on the diagonal over a field K. We assume that K has zero characteristic. The problem of classification of the adjoint and coadjoint orbits of N is far from its solution up today. In the talk we present a complete description of some families of adjoint and coadjoint orbits.

Here we concern the only one aspect of the talk: the description of coadjoint orbits associated with involutions. Class of considered orbits contains all regular orbits and some subregular orbits.

Let $\mathfrak{n}$ be the Lie algebra of N. Let σ be an involution in the symmetric group S_n (i.e. $\sigma^2 = \mathrm{id}$). The involution σ decomposes in the product of commuting reflections $\sigma = r_1 r_2 \cdots r_s$, where r_m is a reflection with respect to the positive root $\xi_m = \varepsilon_{j_m} - \varepsilon_{i_m},\ j_m < i_m$.

Let $\{y_{ij}\}_{1\le j<i\le n-1}$ be a standard basis in $\mathfrak{n}$. Consider the subset X_σ in $\mathfrak{n}^*$ that consists of all $f \in \mathfrak{n}^*$ such that $f(y_{i_m,j_m}) \ne 0$, $1 \le m \le s$, and f annihilates on the other vectors of the standard basis.

THEOREM 1. For any $f \in X_\sigma$ the dimension of the coadjoint orbit $\Omega(f)$ equals to $l(\sigma)-s(\sigma)$ where $l(\sigma)$ is a number of inversions in the permutation $(\sigma(1), \ldots, \sigma(n))$ and $s(\sigma) = s$.

A polarization of $f \in \mathfrak{n}^*$ is a subalgebra that is also a maximal isotropic subspace with respect to the skew symmetric bilinear form $f([x, y])$. It is known that any linear form on a nilpotent Lie algebra has a polarization. A polarization enables to construct a primitive ideal in $U(\mathfrak{n})$ and in the case of $K = \mathbb{R}$ an irreducible unitary representation of N. Our goal is to present a polarization of any $f \in X_\sigma$.

For any $1 \le t \le n$ we consider the involution σ_{t-1} that equals to a product of all reflections r_m, $1 \le m \le s$ such that $j_m < t$. Put $\sigma_0 = \mathrm{id}$. Consider the set of pairs $P_\sigma = \{(i, t) :\ 1 \le t < i \le n,\ \sigma_{t-1}(t) < \sigma_{t-1}(i)\}$. Denote by $\mathfrak{p}_\sigma$ the linear subspace spanned by y_{ij}, $(i, j) \in P_\sigma$.

THEOREM 2. The linear subspace $\mathfrak{p}_\sigma$ is a polarization of any $f \in X_\sigma$.

Construct the symbolic matrix Φ filled by the elements y_{ij}, $i > j$ and zeroes on and upper the diagonal. Consider the characteristic matrix $\Phi(\tau) = \tau\Phi + E$.

For any pair $1 \le k, t \le n$ we consider the ordered systems $J'(k, t) = \mathrm{ord}\{1 \le j < t :\ \sigma(j) > k\}$ and $I'(k, t) = \mathrm{ord}\{\sigma J'(k, t)\}$. Complement $J'(k, t)$ and $I'(k, t)$ to the ordered systems $J(k, t) = J'(k, t) \sqcup \{t\}$ and $I(k, t) = \{k\} \sqcup I'(k, t)$. By $D_{k,t}$ (resp. $D_{k,t}(\tau)$) we denote a minor of the matrix Φ (resp. $\Phi(\tau)$) with the system of columns $J(k, t)$ and system of rows $I(k, t)$.

For any positive root $\zeta = \varepsilon_t - \varepsilon_i$, satisfying $\sigma(\gamma) > 0$, we consider the pair (k, t), where $k = \sigma(i)$. Decompose the minor $D_{k,t}(\tau) = \tau^m(P_{\zeta,0} + P_{\zeta,1}\tau + \ldots)$ where $P_{\zeta,i} \in S(\mathfrak{n}) = K[\mathfrak{n}^*]$ and $P_{\zeta,0} \ne 0$. If $k > t$, then we denote $P_\zeta = P_{\zeta,0}$; if $k < t$, then we denote $P_\zeta = P_{\zeta,1}$. For any $1 \le m \le s$ we denote $D_m = D_{i_m,j_m}$.

THEOREM 3. For any $f \in X_\sigma$ the defining ideal the coadjoint orbit $\Omega(f)$ is generated by P_ζ, where $\zeta \in \Delta^+,\ \sigma(\zeta) > 0$, and $D_m - D_m(f)$, where $1 \le m \le s$.