M. S. Raghunathan
Tata Institute of Fundamental Research
Mumbai, India
msr@math.tifr.res.in

On the first Betti number of compact locally symmetric spaces

This talk is a survey about the first Betti number of a compact locally symmetric space X. We assume that the universal covering $\widetilde{X}$ of X is a product of irreducible of Riemanian symmetric spaces each of which has non-positive sectional curvature and is not flat. We further assume that no finite covering of X breaks up into a non-trivial Riemanian product. When $\widetilde{X}$ is the upper half plane X is a surface of constant negative curvature and detailed information about $H^1(X, \mathbb{Z})$ is classical. Y. Matsushima studied first the case when all factors of X have rank (=dimension of a maximal totally geodesic flat sub manifold) ≥ 2. His results were later completed by Kaneyuki and Nagano to show that $H^1(X, \mathbb{R}) = 0$ in these cases. In the meanwhile Kazhdan came up with a very different spectacular approach to prove that $H^1(X, \mathbb{R}) = 0$ under the above hypothesis on the rank but with the weaker assumption that X has finite volume. Kazhdan made use of representation theory. Later Margulis proved a vanishing theorem when $\widetilde{X}$ has at least two factors. Matsushima provided a very different kind of link of the Betti number problem with representation theory and that has been the basis for all further developments. Kazhdan's ideas were extended to conclude that the only cases where $H^1(X, \mathbb{R}) \neq 0$ are the cases when $\widetilde{X}$ is the hyperbolic n-space or the complex hyperbolic n-space. One expects the following result to hold in general:

If $\widetilde{X}$ is the unit disc in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the hyperbolic or complex hyperbolic metric as the case may be, then X has a finite covering $H^1(X, \mathbb{R}) = 0$.

This expectation has proved to be correct in a large number of cases, the first results being due to Kazhdan. In this talk we outline what is known with broad indications of the ideas behind the proofs. We end the talk with suggesting a possible approach to handle the what seems to be most intractable case: the case when $\widetilde{X}$ is the hyperbolic 3-space. The expected result in this was conjectured by W. Thurston.