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Opers with irregular singularity and spectra of the shift of argument subalgebra

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Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and $U(\mathfrak{g})$ its universal enveloping algebra. The algebra $U(\mathfrak{g})$ bears a natural filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ by the Poincaré–Birkhoff–Witt theorem. The commutator on $U(\mathfrak{g})$ defines the Poisson bracket on $S(\mathfrak{g})$.

Let $ZS(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$ be the center of $S(\mathfrak{g})$ with respect to the Poisson bracket, and let $\mu \in \mathfrak{g}^* = \mathfrak{g}$ be a regular semisimple element. Due to the result of Mischenko and Fomenko (1978) the algebra $\overline{\mathcal{A}_\mu} \subset S(\mathfrak{g})$ generated by the elements $\partial_\mu^n \Phi$, where $\Phi \in ZS(\mathfrak{g})$, is commutative with respect to the Poisson bracket, and has maximal possible transcendence degree equal to $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$. If μ is a regular element contained in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{h} \subset \overline{\mathcal{A}_\mu}$, and if $\mu = f$ is the principal nilpotent element then the subalgebra $\overline{\mathcal{A}_f}$ contains the principal nilpotent subalgebra $\mathfrak{z}_{\mathfrak{g}}(f)$.

It was recently shown that the shift of argument subalgebras can be quantized:

Fact 1. *(R., Feigin – Frenkel – Toledano Laredo) There exist a family of commutative subalgebras $\mathcal{A}_\mu \subset U(\mathfrak{g})$ (where $\mu \in \mathfrak{g}_{reg}$) such that $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}_\mu}$.*

The subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$ is the image of some homomorphism $Z(\hat{\mathfrak{g}}) \to U(\mathfrak{g})$, where $Z(\hat{\mathfrak{g}})$ is the center of the completed enveloping algebra of $\hat{\mathfrak{g}}$ at the critical level.

The main question we discuss is to describe the spectra of $\mathcal{A}_\mu$ in finite-dimensional irreducible $\mathfrak{g}$-modules. The most interesting case is the "most special" case when $\mu = f$ is the principal nilpotent element.

The *principal gradation* on $U(\mathfrak{g})$ is defined on the generators as follows.

$$\deg_{pr} e_\alpha = -(\rho, \alpha^\vee) \ \forall \alpha \in \Delta, \quad \deg_{pr} h = 0 \ \forall h \in \mathfrak{h}.$$

The generators of $\mathcal{A}_f$ are homogeneous with respect to this gradation. Note that the Poincaré series of $\mathcal{A}_f$ with respect to the principal gradation is equal to that of the algebra $U(\mathfrak{n}_-)$. I shall discuss the following main

Theorem 1. *(Feigin – Frenkel – R.) For any integral dominant weight λ the highest vector of V_λ is a cyclic vector for $\mathcal{A}_f$ acting on V_λ. Thus the space V_λ is naturally identified with a quotient of $\mathcal{A}_f$ by a certain ideal $I_\lambda \subset \mathcal{A}_f$. $\mathcal{A}_f/I_\lambda$ is a complete intersection.*

The spectrum of the center at the critical level $Z(\hat{\mathfrak{g}})$ is identified with the space of ${}^LG$-opers on the punctured formal disc (where ${}^LG$ is the Langlands dual group for G). ${}^LG$-opers are connections in the principal G^L-bundle satisfying a certain transversality condition. Namely, for a curve $U = \operatorname{Spec} R$ and some coordinate t on U, the space $\operatorname{Op}_{{}^LG}(U)$ of ${}^LG$-opers is the quotient of the space of ${}^LG$-connections of the form

$$d + (p_{-1} + \mathbf{v}(t))dt, \qquad \mathbf{v}(t) \in {}^L\mathfrak{b}(R)$$

by the action of the group ${}^LN(R)$, where ${}^LN \subset {}^LG$ is the maximal unipotent subgroup, and ${}^L\mathfrak{b} \subset {}^L\mathfrak{g}$ is the Borel subalgebra. For $G = GL_r$, we have $G^L = GL_r$, and G^L-opers are simply differential operators of the degree r.

Since $\mathcal{A}_\mu$ is a quotient of $Z(\hat{\mathfrak{g}})$, the algebra $\mathcal{A}_\mu$ is identified with the algebra of polynomial functions on a certain space of opers. It is shown by Feigin, Frenkel and Toledano Laredo that $\operatorname{Spec} \mathcal{A}_\mu$ is the set of ${}^LG$-opers, which

1. are defined globally on $\mathbb{C}P^1 \backslash \{0, \infty\}$,
2. have regular singularity at 0
3. have an irregular singularity of the degree 2 at ∞ with the 2-residue μ.

Moreover, the image of $\mathcal{A}_\mu$ in $\mathrm{End}(V_\lambda)$ factors through the opers, which

1. have the residue λ at 0;
2. have trivial monodromy representation.

For every dominant weight λ, the no-monodromy condition of corresponding ${}^L G$-opers is a finite set of polynomial relations in the generators of $\mathcal{A}_f$. The number of such relations is equal to the number of positive roots of $\mathfrak{g}$. These relations have the degrees $(\alpha^\vee, \lambda + \rho)$ with respect to the principal grading. We prove that the no-monodromy conditions generate the ideal I_λ.

Note that this agrees with the q-analog of the Weyl dimension formula. Namely, the Poincaré series of any irreducible finite-dimensional $\mathfrak{g}$-module V_λ with respect to the principal grading is

$$\chi_\lambda(q) = \prod_{\alpha > 0} \frac{1 - q^{(\alpha^\vee, \lambda + \rho)}}{1 - q^{(\alpha^\vee, \rho)}}.$$

We note that the non-central generators of $\mathcal{A}_f$ have the degrees $(\alpha^\vee, \rho)$ with respect to the principal grading, and the no-monodromy relations have the degrees $(\alpha^\vee, \lambda + \rho)$, and hence $\mathcal{A}_f / I_\lambda$ has the same Poincaré series.