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Geometric methods for semi-invariants of quivers

For a quiver Q with n vertices and a dimension vector $\alpha \in \mathbf{Z}_+^n$ the set $R(Q,\alpha)$ of representations of Q with dimension α is interesting for both Representation and Invariant Theories. From the point of view of Invariant Theory, $R(Q,\alpha)$ is a module over the reductive group $GL(\alpha)$ and the natural questions about the action $GL(\alpha) : R(Q,\alpha)$ are the orbits, the orbit closures, the invariants etc. For example, the invariants are in general closely related with the closed orbits, and the orbit of a representation V is closed if and only if V is semi-simple and in this case the slice étale at V is described in terms of a local quiver Σ_V, which is defined from the indecomposable summands of V (see [LBP]).

However, in many ineresting cases all regular invariants are constant and the only semi-simple representation is the zero point of $R(Q,\alpha)$. In all cases we have a more general object, the regular semi-invariants of $GL(\alpha)$ or the invariants of $SL(\alpha)$, the commutator subgroup in $GL(\alpha)$. The module $\mathbf{k}[R(Q,\alpha)]_\sigma^{(GL(\alpha))}$ of semi-invariants of weight σ has a nice description in terms of Schofield's determinantal semi-invariants and perpendicular categories ${}^\perp W$ and $W^\perp$ (see [Sch]). Using these ideas, we suggested a geometical approach to semi-invariants based on:

Theorem 1. *([Sh1]) Let $V = m_1S_1 + \cdots + m_tS_t \in R(Q,\alpha)$ be a decomposition into indecomposable summands. The following properties of V are equivalent:*
(i) the $SL(\alpha)$-orbit of V is closed in $R(Q,\alpha)$
(ii) the $GL(\alpha)$-orbit of V is closed in $R(Q,\alpha)_f$, $f \in \mathbf{k}[R(Q,\alpha)]_\sigma^{(GL(\alpha))}$
(iii) $S_1,\cdots,S_t$ are simple objects in ${}^\perp W$ for a representation W.

We call the representations fulfilling the above equivalent conditions *locally semi-simple.* The condition (ii) of the Theorem allows to apply Luna étale slice Theorem [Lu], and the slice at V is described by the local quiver Σ_V exactly in the same style as in semi-simple case. Luna slice étale Theorem gives rise to a stratification of the quotient $R(Q,\alpha)/\!\!/SL(\alpha)$ by the locally-closed strata $(R(Q,\alpha)/\!\!/SL(\alpha))_{(H)}^{GL(\alpha)}$, such that the stabilizers in $GL(\alpha)$ of the points in the closed orbit in the fiber are conjugate to $H \subseteq GL(\alpha)$. The locally semi-simple representations over a stratum have the same dimensions of indecomposable summands, so the strata are in 1-to-1 correspondance with the *locally semi-simple decompositions* of α and, similarly to the usual Luna stratification, there is a unique *generic* locally semi-simple decomposition. This generic stabilizer H gives rise to a version of the Luna-Richardson Theorem relating the semi-invariants of an affine action $G : X$ with those for $N_G(H)$ acting on X^H, and this was applied in [Sh1] to the semi-invariants of tame quivers.

So it is useful to calculate locally semi-simple decompositions, in particular, generic. Derksen and Weyman introduced in [DW] the notion of *quiver Schur sequence*, a sequence $\underline{\beta} = (\beta_1,\cdots,\beta_k)$ of Schur roots with special properties.

Theorem 2. *[Sh2] A decomposition with the summands from a quiver Schur sequence is locally semi-simple.*

When the dimension vector α is *prehomogeneous*, i.e., $GL(\alpha)$ has a dense orbit in $R(Q,\alpha)$, then the converse to the last Theorem is true and moreover

Theorem 3. *[Sh2] If α is prehomogeneous, then there is an order preserving isomorphism of the strata $(R(Q,\alpha)/\!\!/SL(\alpha))_{(H)}^{GL(\alpha)}$ with the faces of a simplicial cone.*

In [Sh2] we presented an algorithm for calculating the generic locally semi-simple decomposition for a quiver without oriented cycles and arbitrary dimension vector. This algorithm (and some other related ones) was implemented in the computer program TETIVA (see [Te]).

References

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