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# Real elements in algebraic groups

Let $G$ be an algebraic group defined over a field $k$. We call an element $g \in G$ **real** if $g$ is conjugate to $g^{-1}$ in $G$. We say $g \in G(k)$ is **$k$-real** if there exists $h \in G(k)$ such that $hgh^{-1} = g^{-1}$. Note that every element in the conjugacy class of a real element $g$ is real. Such conjugacy classes are called real. An element $t \in G$ is called an **involution** if $t^2 = 1$. If an involution in $G$ conjugates $g$ to $g^{-1}$, then it follows that $g$ is a product of two involutions in $G$ and conversely, any such element is real. An element $g \in G(k)$ is called **strongly real** if $g$ is a product of two involutions in $G(k)$.

In [ST1] and [ST2] , we deal with results concerning real elements in algebraic groups, defined over an arbitrary field. Let $G$ be a connected semisimple algebraic group of adjoint type defined over a perfect field $k$. Suppose the longest element $w_0$ of the Weyl group $W(G,T)$ acts by $-1$ on the set of roots with respect to a fixed maximal torus $T$. Then for a strongly regular element $t \in G(k)$, we prove that $t$ is real in $G(k)$ if and only if $t$ is strongly real in $G(k)$ ([ST2], Theorem 2.1.2). Moreover, we prove that every element of a maximal torus, containing a real strongly regular element, is strongly real. We study the structure of real semisimple elements in groups over fields with $cd(k) \leq 1$. Let $k$ be such a field. Let $G$ be a connected reductive group defined over $k$. Then, semisimple elements in $G(k)$ are real in $G(k)$ ([ST2], Theorem 2.3.3). It follows that if $G$ is connected semisimple of adjoint type, with $-1$ in its Weyl group, then every semisimple element in $G(k)$ is strongly real in $G(k)$. This also shows that any regular element in such a group is real.

In [TZ], Tiep and Zalesski prove that in a simple algebraic group $G$ over an algebraically closed field, all elements of $G$ are real if and only if $G$ is of type $B_n, C_n, D_{2n}, G_2, F_4, E_7$ or $E_8$. In [ST2], with case by case consideration, we prove the following result. Let $G$ be an algebraic group defined over a perfect field $k$ of characteristic $\neq 2$. Let $t$ be an element of $G$ of one of the following type: Any element in $GL_n(k), SL_n(k) (n \not\equiv 2 \pmod 4)$ or groups of type $G_2$ defined over $k$ or a semisimple element in $PSL_1(Q)$ ($Q$, a quaternion algebra), $O(q), SO(q) (n \not\equiv 2 \pmod 4$ and $q$ a nondegenerate quadratic form) or $PSp(2n)$. Then $t$ is real in $G$ if and only if $t$ is strongly real.

Following the above result in [ST2] more closely we get structure of a real element as follows. Let $k$ be a perfect field of characteristic $\neq 2$. Let $t$ be an element of $G$ of one of the following type: any element in $GL_n(k), SL_n(k)$ or groups of type $G_2$ defined over $k$ or a semisimple element in groups of type $A_1$, $O(q), SO(q)$ or $Sp(2n)$ where $q$ is a nondegenerate quadratic form over $k$. Then $t$ is real in $G$ if and only if $t$ has a decomposition $t = \tau_1\tau_2$ where $\tau_1, \tau_2 \in G$ and $\tau_1^2 = \pm 1 = \tau_2^2$. In [S], we investigate the structure of real semisimple elements in Spin groups and prove similar results when dimension is $0, 1$ or $2 \pmod 4$.

In finite group theory, the number of real irreducible characters, which come from either orthogonal or symplectic representations, is same as the number of real conjugacy classes. Our results, combined with those in [Pr1], [Pr2], suggest a relation between strongly real classes in groups with their orthogonal representations. This is yet to be explored. The finite simple groups which have all elements strongly real are classified in [KN] and [TZ].

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