I. D. Suprunenko
Institute of Mathematics, National Academy of Sciences of Belarus
Minsk, Belarus
suprunenko@im.bas-net.by

Lower estimates for the maximal weight multiplicities in irreducible representations of the classical algebraic groups

The goal of this talk is to discuss some results on lower estimates for the maximal weight multiplicities in irreducible representations of the classical algebraic groups in positive characteristic obtained in a course of a joint research with A.A. Baranov and A.A. Osinovskaya. For the groups of types B_n, C_n and D_n such estimates are found under some mild restrictions on the ground field characteristic that do not depend upon n. For the groups of type A_n at present relevant estimates are obtained only for n large enough with respect to the characteristic.

In what follows K is an algebraically closed field of characteristic $p > 0$; G is a classical algebraic group of rank n over K; Irr is the set of all rational irreducible representations (or simple modules) of G up to equivalence, $\mathrm{Irr}_p \subset \mathrm{Irr}$ is the subset of p-restricted ones; $\omega(M)$ is the highest weight of a simple module M, $\mathrm{wdeg}\, M$ is the maximal dimension of a weight subspace in M; $\omega_1 \ldots, \omega_n$ are the fundamental weights of G labeled in a standard way. Recall that a weight $\sum_{i=1}^n a_i\omega_i$ is p-restricted if $0 \le a_i < p$ for all i.

For the classical algebraic groups the simple modules M with $\mathrm{wdeg}\, M = 1$ were classified in [1, 3]. That result was used in the description of the maximal subgroups of such groups in [1]. Denote by Ω the set of dominant weights ω of G such that $\mathrm{wdeg}\, M = 1$ for a simple G-module M with highest weight ω and by Ω_p the subset of all p-restricted weights in Ω. Assume that $p \neq 2$ for $G \neq A_n(K)$. Then for $n \geq 4$

$$\Omega_p = \begin{cases} \{0; \omega_k, 1 \le k \le n; \quad (p-1-a)\omega_k + a\omega_{k+1}, 0 \le k \le n, 0 \le a \le p-1\} & \\ & \text{for } G = A_n(K), \\ \{0, \omega_1, \omega_n\} & \text{for } G = B_n(K), \\ \{0, \omega_1, \frac{p-1}{2}\omega_n, \omega_{n-1} + \frac{p-3}{2}\omega_n\} & \text{for } G = C_n(K), \\ \{0, \omega_1, \omega_{n-1}, \omega_n\} & \text{for } G = D_n(K), \end{cases}$$

and the weight $\lambda = \sum_{j=0}^s p^j \lambda_j$ with p-restricted λ_j lies in Ω if and only if all $\lambda_j \in \Omega_p$. Here for $G = A_n(K)$ assume that $\omega_0 = \omega_{n+1} = 0$.

Theorem 1. *Let $n = 4k + r \geq 8$ with integer k and r and $0 \le r < 4$. Assume that $G = B_n(K)$, $C_n(K)$ or $D_n(K)$, $M \in \mathrm{Irr}$ and $\omega(M) \notin \Omega$. Suppose that $p > 2$ for $G = B_n(K)$ or $D_n(K)$ and $p > 7$ for $G = C_n(K)$. Then $\mathrm{wdeg}\, M \geq n - 4 - r \geq n - 7$.*

One easily concludes that $\mathrm{wdeg}\, M = n$ if $p > 2$, $n > 2$ for $G = B_n(K)$ and > 3 for $G = D_n(K)$, $M \in \mathrm{Irr}$ and $\omega(M) = \omega_2$ for $G = B_n(K)$ or $D_n(K)$ and $2\omega_1$ for $G = C_n(K)$. Hence the estimates in Theorem 1 are asymptotically exact.

For groups of type A_n the situation is substantially more difficult. For each positive integer a and n large enough with respect to a there exist p-restricted $A_n(K)$-modules M with $\mathrm{wdeg}\, M > a$, but small enough with respect to n. Indeed, if $\omega(M) = \sum_{i=1}^d a_i\omega_i$ or $\sum_{i=1}^d a_i\omega_{n-i+1}$, then for large n the parameter $\mathrm{wdeg}\, M$ is bounded by some constant that depends on $a_1, \ldots, a_d$ only and does not depend on n. For this type one has to take into account the polynomial degree of an irreducible module M when trying to estimate $\mathrm{wdeg}\, M$. If $\omega(M) = \sum_{i=1}^n a_i\omega_i$, set $\mathrm{pdeg}\, M = \sum_{i=1}^n ia_i$. Below M^* is the module dual to M.

Theorem 2. *Let $G = A_n(K)$, $M \in \mathrm{Irr}_p$, $M \notin \Omega_p$, and $n > 4p^2$. Assume that $\mathrm{pdeg}\, M > n$ and $\mathrm{pdeg}\, M^* > n$. Then $\mathrm{wdeg}\, M > \sqrt{n}/p - 1$.*

For $G = A_n(K)$ we also suppose that a stronger (may be, linear) estimate holds for wdeg M if both pdeg M and pdeg $M^* > n$. Some results on wdeg M for tensor decomposable irreducible modules M will be discussed as well.

Estimates of weight multiplicities indicated above and expected results in this direction for groups of type A_n can be used for recognizing linear groups containing matrices with small eigenvalue multiplicities. Indeed, for groups of types B_n, C_n, and D_n it occurs (may be, under some restrictions on the characteristic) that the images of almost all their representations do not contain matrices all whose eigenvalue multiplicities are small enough with respect to the group rank, only the representations lying in certain well-defined classes yield exceptions.

These estimates enable us to classify inductive systems of representations with totally bounded weight multiplicities for natural embeddings of classical groups. Inductive systems of representations have been introduced by A.E. Zalesskii in [2]. They yield an asymptotic version of the branching rules for relevant embeddings and can be applied to the study of ideals in group algebras of locally finite groups.

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