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Calculations in exceptional groups

Let Φ be a reduced irreducible root system, R be a commutative ring with 1. We study the following three closely related groups, associated to (Φ, R).

- The (simply-connected) Chevalley group $G(\Phi, R)$.
- The (simply-connected) elementary Chevalley group $E(\Phi, R)$.
- The Steinberg group $\mathrm{St}(\Phi, R)$.

We set $K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R)$ and denote by $K_2(\Phi, R)$ the kernel of the natural projection $\mathrm{St}(\Phi, R) \longrightarrow E(\Phi, R)$. We are mainly interested in the four large exceptional groups of types E_6, E_7, E_8 and F_4.

There are two approaches towards the proof of structure theorems for $G = G(\Phi, R)$: induction on dimension of R (localisation proofs) and induction on rank of Φ (geometric proofs).

Using Bak's method of localisation-completion, A.Bak, R.Hazrat and the author [1], [2] succeeded in finalising results on the nilpotent structure of $K_1(\Phi, R)$. For example, (apart from the known exceptions of rank ≤ 2) we characterised $E(\Phi, R)$ as the unique largest perfect subgroup of $G(\Phi, R)$ and, among other things, proved the following results. Recall, that $\delta(R)$ denotes the Bass—Serre dimension of R.

Theorem 1. *Assume, that* $\mathrm{rk}(\Phi) \geq 2$, *and* $\Phi \neq \mathrm{B}_2, \mathrm{G}_2$. *Then the subgroup* $E(\Phi, R, A, B)$ *is normal in* $G(\Phi, R)$, *while* $C(\Phi, R, A, B)$ *coincides with the transporter* $\mathrm{Tran}(G(\Phi, R), E(\Phi, R, A, B))$.

Theorem 2. *Assume, that* $\mathrm{rk}(\Phi) \geq 2$, *and* $\delta(R) < \infty$. *Then the group* $\mathrm{K}_1(\Phi, R, I)$ *is nilpotent.*

However, our main emphasis is on calculations in exceptional groups, based on geometry and combinatorics of minimal modules. We wish to calculate with elements of E_6, E_7, E_8 as 27×27, 56×56 or 248×248 matrices, respectively. Straightforward calculations with such matrices, using equations of degree ≥ 3 immediately run into formidable difficulties.

Developing pivotal ideas of H.Matsumoto and M.Stein, the author, A.Stepanov and E.Plotkin [3] – [5], [11] proposed the first *working* method of such calculations, DECOMPOSITION OF UNIPOTENTS, which only referred to *quadratic* equations on one column or row of a matrix from G.

Recently the author and his students M.Gavrilovich, S.Nikolenko and A.Luzgarev, see, in particular, [6] – [10] developed another amazing geometric approach to the proof of the main structure theorems for $G(\Phi, R)$, PROOF FROM THE BOOK aka the A_2-proof, which only refers to *linear* equations on the Lie algebra of G.

In the A_2-proof we look at the stabiliser of a column of a matrix of the form $e + x$, where x is an element of the corresponding Lie algebra L. *In essense* what we prove in [7], [8], [10] is the following result.

Theorem 3. *Let* $z \in G$ *and* $g = [z, x_\delta(1)]$ *is non-central. Then there exists a root type unipotent* $x = x_{\beta_1}(\xi)x_{\beta_2}(\zeta)$ *such that* $(xg)_{*\omega} = g_{*\omega}$ *and* $[x, g] \neq e$.

This easy observation immediately implies many known structure results about Chevalley groups, *including* description of their normal subgroups. We observe that this trick works for Chevalley groups of types E_6 and E_7 in minimal representations [6], not for two arbitrary columns, of course, not even for any two singular columns at distance 1, but for two columns of a root type inipotent at distance 1.

Theorem 4. *Let $z \in G$ and $g = [z, x_\delta(1)]$ is non-central. Then there exists a root type unipotent $x = x_{\beta_1}(\xi)x_{\beta_2}(\zeta)x_{\beta_3}(\eta)$ such, that $(xg)_{*\lambda} = g_{*\lambda}$, $(xg)_{*\mu} = g_{*\mu}$ for some weights λ, μ at distance 1 and $[x, g] \neq e$.*

We state also some other closely related results, such as the main lemma of the A_3-proof, proposed by the author to assault centrality of $K_2(\Phi, R)$ for exceptional groups [6]; a characterisation of the extended Chevalley group of type E_6 over an arbitrary commutative ring, found recently by the author and A.Luzgarev [9]; and finiteness of commutators in elementary generators, established by A.Stepanov and the author.

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