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Volumes and complexity of hyperbolic three-dimensional manifolds

If M is a compact 3-dimensional manifold, its *complexity* [1] is a non-negative integer $c(M)$ which formally translates the intuitive notion of "how complicated" M is. In particular, if M is closed and irreducible and different from the 3-sphere $\mathbb{S}^3$, the projective 3-space $\mathbb{RP}^3$, and the lens space $L(3,1)$, its complexity $c(M)$ is equal to the minimum of the number of tetrahedra over all triangulations of M. The complexity function has many natural properties, among which additivity under connected sum.

The task of computing the complexity $c(M)$ of a given manifold M is extremely difficult. For closed M, the exact value is presently known only if M belongs to the computer-generated tables of manifolds up to complexity 12, see [2].

We will present recent results obtained jointly with Sergei Matveev and Carlo Petronio on two-sided bounds for complexity for some infinite families of closed orientable hyperbolic 3-manifolds. In all cases the upper bounds are based on triangulating of fundamental polyhedra and the low bounds are based on computations or estimations of hyperbolic volumes.

The first family consists of Löbell manifolds $\mathcal{L}(n)$ which generalize the first example of a closed orientable hyperbolic 3-manifold constructed by F. Löbell [3] in 1931.

Theorem 1 ([4],[5]). *If $\mathcal{L}(n)$ denotes the n-th set of Löbell manifolds, then for any $M \in \mathcal{L}(n)$ the complexity of M satisfies the lower and upper estimates*

$$10n \leqslant c(M) \leqslant 47n - 92,$$

the former one being valid for sufficiently large n and the latter one for all $n \geqslant 5$.

Here the crucial point for the lower estimate is the Vinberg formula for volume of a double-rectangular tetrahedron [6].

The second family of manifolds consists of cyclic branched coverings of the 3-sphere branched along two-bridge links. For a two-bridge link $K(p,q)$ let us denote by $\ell(p,q)$ the minimum of lengths of continuous fractions representing the rational number p/q.

Theorem 2 ([7]). *Let $K(p,q)$ be a given two-bridge knot (or link) and let $(M_n(p,q))_{n=2}^{\infty}$ be a sequence of meridian-cyclic n-fold branched coverings of S^3, branched along $K(p,q)$. Then:*

$$c(M_n(p,q)) \leqslant n(p-1) \quad \forall n.$$

If in addition $K(p,q)$ is hyperbolic then the following inequality holds for $n \geqslant 7$ with $c = 4$:

$$c(M_n(p,q)) > n \cdot \left(1 - \frac{c\pi^2}{n^2}\right)^{3/2} \cdot \max\{2, 2\ell(p,q) - 2.6667\ldots\};$$

moreover, if $K(p,q)$ is neither $K(5,2)$ nor $K(7,3)$, then the inequality holds for $n \geqslant 6$ with $c = 2\sqrt{2}$.

We will also discuss some particular cases of 2-bridge knots for which the estimates can be improved. For example, for the Fibonacci manifold $M(n)$ constructed in [8], which is the n-fold cyclic branched covering of S^3 branched along the figure-eight knot, we get

Theorem 3 ([5]). *If $M(n)$ denotes the n-th Fibonacci manifold, then its complexity satisfies the lower and upper estimates*

$$2n \leqslant c(M(n)) \leqslant 3n,$$

the former one being valid for sufficiently large n and the latter one for all $n \geqslant 4$.

References

[1] S. V. Matveev, *Algorithmic topology and classification of* 3*-manifolds*, ACM-monographs Vol. **9**, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[2] S. V. Matveev, *Recognition and tabulation of* 3*-manifolds*, Dokl. Akad. Nauk **71** (2005), 20-22.

[3] F. Löbell, *Beispiele geschlossene dreidimensionaler Clifford — Kleinischer Räume negative Krümmung*, Ber. Verh. Sächs. Akad. Lpz., Math.-Phys. Kl. **83** (1931), 168-174.

[4] S. Matveev, C. Petronio, A. Vesnin, *Two-sided complexity bounds for Löbell manifolds*, Dokl. Akad. Nauk **76** (2007), 689-691.

[5] S. Matveev, C. Petronio, A. Vesnin, *Two-sided asymptotic bounds for the complexity of some closed hyperbolic three-manifolds*, Preprint math.GT/0602372.

[6] E. Vinberg, *Volumes of non-Euclidean polyhedra*, Russian Math. Surveys **48(2)** (1993), 15–45.

[7] C. Petronio, A. Vesnin, *Two-sided asymptotic bounds for the complexity of cyclic branched coverings of two-bridge links*, Preprint math.GT/0612830.

[8] H. Helling, A. C. Kim, J. L. Mennicke, *A geometric study of Fibonacci groups*, J. Lie Theory **8** (1999), 1-23.