V. E. Voskresenskii
Samara State University
Samara, Russia
voskres@ssu.samara.ru

Birational geometry of algebraic tori

Let k be a field of zero characteristic, $\bar{k}$ its algebraic closure, and Π the Galois group of the extension $\bar{k}/k$. Let X be an algebraic variety over k and $\bar{X} = X \otimes_k \bar{k}$. The group Π acts naturally on $\bar{X}$ and on objects defined by the scheme $\bar{X}$. The k-variety Y is called a k-form of X if varieties $\bar{X}$ and $\bar{Y}$ are isomorphic over the field $\bar{k}$. If X is quasiprojective over k, then the set $H^1(\Pi, Aut\bar{X})$ describes the set of classes of all k-forms of X. Two Π-modules A and B will be called similar if there exist two permutations Π-modules S_1 and S_2 such that $A \oplus S_1 \cong B \oplus S_2$. The similarity class of a Π-module A is denoted by $[A]$. The following assertion is known. Let X and Y be k-birationally equivalent nonsingular projective varieties over field k. Then the Π-modules $\mathrm{Pic}\bar{X}$ and $\mathrm{Pic}\bar{Y}$ are similar and groups $H^1(\Pi, \mathrm{Pic}\bar{X})$ and $H^{-1}(\Pi, \mathrm{Pic}\bar{X})$ are birational invariants of X.

Let G be a connected linear algebraic group over k and $\hat{G}$ be the Π-module of rational characters of G. We consider a nonsingular projective variety X over k that contains G as an open subset. The variety X is called a projective model of the group G. The class $[\mathrm{Pic}\bar{X}] = p(G)$ is a birational invariant of G. The embedding $G \subset X$ induces an exact sequence of modules $0 \to \hat{G} \to \hat{S} \to \mathrm{Pic}\bar{X} \to \mathrm{Pic}\bar{G} \to 0$. If $G = T$ is an algebraic torus then $\mathrm{Pic}\bar{T} = 0$ and we have an exact sequence of torsion-free modules $0 \to \hat{T} \to \hat{S} \to \mathrm{Pic}X_L \to 0$, where L is a splitting field of $T, \Pi = Gal(L/k), (L : k) < \infty$, and $\hat{S}$ is a permutation Π-module. We have an unexpected result which is one of the most important results in birational classification of algebraic tori: $H^{-1}(L/F, \mathrm{Pic}X_L) = 0, k \subset F \subset L$. Now we can give a purely algebraic description of the birational invariants $[\mathrm{Pic}X_L]$. For algebraic tori there is a sufficiently simple, one can say, canonical construction of complete models suggested by M. Demazure. A complete toric variety is constructed by gluing affine toric varieties, and the scheme of gluing is determined by some set of cones called a fan. As one may expect, tori T with $p(T) = [0]$ have some special features. Such a torus T can be described as the following factor-group: $1 \to S_1 \to S_2 \to T \to 1$, where S_1 and S_2 are quasisplit k-tori. We have just obtained the main result of this article.

Theorem. A*ny stably rational k-torus is rational over k.*

References

[1] V. E. Voskresenskii. Algebraic Groups and Their Birational Invariants, Translation of Mathematical Monographs, vol.179, American Mathematical Society, Providence, 1998.