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On symmetric invariants of $\mathbb{Z}_2$-degenerations

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Suppose that G is a simply connected reductive algebraic group defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. Set $\mathfrak{g} = \operatorname{Lie} G$ and let r denote the rank of this Lie algebra. Let σ be an involution of $\mathfrak{g}$. Then it defines a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 := \mathfrak{g}^\sigma$ is a symmetric subalgebra of $\mathfrak{g}$. The corresponding $\mathbb{Z}_2$-*degeneration* of $\mathfrak{g}$ is the Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is considered as a commutative ideal. It was conjectured by Panyushev that the algebra of symmetric invariants $\mathcal{S}(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{g}}}$ is polynomial. The conjecture is true and we will present basic ideas of a proof.

First we need some basic fact concerning semidirect rpoducts and their invariants. Let $\mathfrak{q}$ be an arbitrary Lie algebra and V a space of a finite dimensional representation of $\mathfrak{q}$. Then one form a semidirect products $\mathfrak{q} \ltimes V$, where V is a commutative ideal. It is possible to define a certain Lie algebra

$$\hat{\mathfrak{q}} := \{\xi \in \mathfrak{q} \otimes \mathbb{K}(V^*) \mid \xi(x) \in \mathfrak{q}_x \text{ for all } \ x \in V^*\}$$

over the field $\mathbb{K}(V^*)$, which is a Lie algebra of rational sections for the bundle $\Gamma = \{y, \mathfrak{q}_y\} \subset V^* \times \mathfrak{q}$. We need a more precise description (and the existence proof) of this Lie algebra. Let us a choose an arbitrary basis $\eta_1, \ldots, \eta_k$, where $k = \dim \mathfrak{q}$, of $\mathfrak{q}$. An element $a_1\eta_1 + \ldots + a_k\eta_n$ with $a_i \in \mathbb{K}(V^*)$ lies in $\hat{\mathfrak{q}}$ if and only if $\sum_{i=1}^k a_i x{\cdot}\eta_i = 0$ for all $x \in V^*$. Here $x{\cdot}\eta_i$ is a linear function of $\mathfrak{q}$ given by $x{\cdot}\eta_i(q) = \eta(\operatorname{ad}^*(q)x)$ for each $q \in \mathfrak{q}$. Choosing a basis $\{x_1, \ldots, x_p\}$ of V^* we get $p = \dim V$ equations defining $\hat{\mathfrak{q}}$ over $\mathbb{K}(V^*)$. Rank of the arising $k \times p$ matrix $M = (m_{ij})$ with $m_{ij} = [\eta_i, x_j]$ is equal to the codimension of a generic Q orbit on V^*. Therefore $\hat{\mathfrak{q}}$ is a linear space (over $\mathbb{K}(V^*)$) of dimension $\dim \mathfrak{q}_x$ (with $x \in V^*$ generic).

Take any $x \in V^*$ and let $\hat{\mathfrak{q}}(x)$ be the image of an obvious "evaluation" homomorphism from $\hat{\mathfrak{q}}$ to $\mathfrak{q}_x$. Set

$$V^*_{\rm ns} := \{x \in V^* \mid \hat{\mathfrak{q}}(x) \neq \mathfrak{q}_x\}.$$

Note that if an orbit $Qx \subset V^*$ is not of maximal dimension, then $x \in V^*_{\rm ns}$.

Lemma 1. *The codimansion of $V^*_{\rm ns}$ in V^* is at least two.*

Set $G_0 := G^\sigma$. Since G is assumed to be simply connected, G_0 is connected. Let L denote a generic stabiliser of the action of G_0 on $\mathfrak{g}_1$ and set $\mathfrak{l} := \operatorname{Lie} L$. Since the action of G_0 on $\mathfrak{g}_1$ is self dual, the group L is reductive. Set $\ell := \operatorname{rk} \mathfrak{l}$. For each point $y \in \mathfrak{g}_i$ (or in $\mathfrak{g}_i^*$) let $\mathfrak{g}_{0,y}$ and $\mathfrak{g}_{1,y}$ denote the stabiliser of y in $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. There is a symmetric decomposition $\mathfrak{g}_y = \mathfrak{g}_{0,y} \oplus \mathfrak{g}_{1,y}$.

Lemma 2. *Let $Y = \mathfrak{g}_0^* \times \{y\}$ be a subset of $\widetilde{\mathfrak{g}}^*$ with $y \in \mathfrak{g}_1^*$. Then $\mathbb{K}[Y]^{\mathfrak{g}_1} \cong \mathbb{K}[\mathfrak{g}^*_{0,y}]$.*

Proof. This is a particular case of [2, Lemma 4]. □

Set $\mathbb{F} := \mathbb{K}(\mathfrak{g}_1^*)^{G_0}$. Clearly $\mathbb{F} \subset \mathbb{K}(\widetilde{\mathfrak{g}})^{\widetilde{G}}$. In our case of interest $\mathbb{F} = \operatorname{Quot} \mathbb{K}[\mathfrak{g}_1^*]^{G_0}$. Over the new field $\mathbb{F}$ the action of G_0 on $\mathfrak{g}_1^*$ is locally transitive. Therefore Lemma 2 yields the following.

Corollary 1. *We have $\mathbb{K}[\widetilde{\mathfrak{g}}^*]^{\widetilde{\mathfrak{g}}} \otimes_{\mathbb{K}[\mathfrak{g}_1^*]^{G_0}} \mathbb{F} \cong \mathbb{F}[\mathfrak{l}^*]^L$ and $\mathbb{K}(\widetilde{\mathfrak{g}})^{\widetilde{G}} \cong \mathbb{K}(\mathfrak{g}_1^*)^{G_0} \otimes \mathbb{K}(\mathfrak{l}^*)^L \cong \mathbb{F}(\mathfrak{l}^*)^L$.*

Theorem 1. *For each $\mathbb{Z}_2$-degeneration $\widetilde{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$, the algebra $\mathcal{S}(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{g}}}$ of symmetric invariants is a Polynomial algebra in r variables.*

Sketch of a proof. Choose a set of homogeneous generators $\{H_1, \ldots, H_\ell\} \subset \mathbb{F}[\mathfrak{l}^*]^L$. Due to Corollary 1 they can be regarded as rational functions on $\widetilde{\mathfrak{g}}^*$. Let $U \subset \mathfrak{g}_1^*$ be the maximal subset such that all H_i are regular on $\mathfrak{g}_0^* \times U$. The complement $\mathfrak{g}_1^* \backslash U$ is a union of G_0-invariant divisors, i.e., a union of zero sets of G_0-invariant regular functions. Therefore, multiplying by dominator, we can make all generators $H_1, \ldots, H_\ell$ regular. New functions also form a set of generators of $\mathbb{K}[\widetilde{\mathfrak{g}}^*]^{\widetilde{\mathfrak{g}}} \otimes_{\mathbb{K}[\mathfrak{g}_1^*]^{G_0}} \mathbb{F}$ over the field $\mathbb{F}$. For that reason we do not change the notation.

For $y \in \mathfrak{g}^*$ let

$$\varphi_y : \ \mathbb{K}[\widetilde{\mathfrak{g}}^*] \to \mathbb{K}[\mathfrak{g}_0^* \times \{y\}]$$

be the restriction homomorphism. The algebra of invariants $\mathcal{S}(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{g}}}$ is free if and only if there is a set of generators $H_1, \ldots, H_\ell$ such that $\varphi_y(H_i)$ are algebraically independent for the elements y of a big open subset of $\mathfrak{g}_1^*$.

To verify this we need to check that each G_0-invariant divisors $X \subset \mathfrak{g}_1*$ contains a point x with $\varphi_x(H_i)$ being algebraically independent.

Making use of Lemma 1, this question can be reduced to simpler question concerning a so called *degeneration* of $\mathfrak{l}$ to $\mathfrak{g}_{0,x}$ with $x \notin (\mathfrak{g}_1^*)_{\rm ns}$.

For several involution Theorem 1 was proved by Panyushev [1]. The last part of our proof is a case by case verification for the remaining ones:

- (E_6, F_4), $(E_7, E_6 \oplus \mathbb{K})$, $(E_8, E_7 \oplus \mathfrak{sl}_2)$, $(E_6, \mathfrak{so}_{10} \oplus \mathbb{K})$, $(E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$;
- $(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m})$, with $n \geqslant m$; $(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$; $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$.

References

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